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Stochastic Modeling for Commodity Prices and Valuation of Commodity Derivatives under Stochastic Convenience Yields and Seasonality

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Abstract

In this dissertation, we develop a two-factor model of the stochastic behavior of commodity prices. The first factor is the commodity spot price which follows a geometric Brownian motion with a time-varying volatility. The second factor is the instantaneous convenience yield which follows an extended Cox-Ingersoll-Ross (CIR) process by adding a time-dependent function into the drift term of the process in order to describe seasonal variations in commodity prices. The time-varying volatilities of the commodity spot prices and the instantaneous convenience yields are proportional to the square root of the instantaneous convenience yields. Our modeling concerns about two important things: a link between price volatilities and convenience yields as suggested by the theory of storage, and the seasonality in commodity prices and convenience yield volatilities. We establish sufficient conditions to guarantee the inaccessibility to nonpositive values of the volatility process. Closed-form solutions for futures prices are derived under the standard no-arbitrage arguments. The closed-form solutions are consistent with the theory of storage: futures prices tend to be lower than spot prices when convenience yields are sufficiently high and vice versa. In addition, the closed-form solutions lead to extraction formulas for the two factors under the assumption that two no-arbitrage futures prices having different maturities can be observed. Moreover, European futures options prices are determined by using a method of Fourier transforms. We estimate the model parameters using the daily futures prices data of two agricultural commodities in Thailand: rice and natural rubber. The futures prices data are obtained from the Agricultural Futures Exchange of Thailand (AFET) in sample periods in August 2004 to August 2006. The estimation method is based on a maximum likelihood approach. The empirical results are in accordance with the theory of storage and we have a comment on the Thai price intervention scheme. Using the estimated parameters, we calculate price differences and correlations between the observed futures prices and the predicted futures prices, obtained from our model, for several futures contracts of the two commodities. The results obtained show the observed and the predicted futures prices are insignificantly different and strongly positive correlated. Finally, we analyze the implications of our model for capital budgeting decisions by investigating the situations known as backwardation[I] and contango[II] in AFET. We have found that, for long maturity futures contracts, the futures market of rice exhibited backwardation, while the futures market of natural rubber exhibited contango.

Keywords: Modeling for commodity prices, stochastic convenience yields, theory of storage, seasonality, futures, futures options, maximum likelihood estimation.

[I, II] See the definitions of "backwardation" and "contango" in Section 3.7 of Chapter 3.

Zusammenfassung

In dieser Dissertation entwickeln wir ein Modell mit zwei Faktoren, welches das stochastische Verhalten von Warenpreisen beschreibt. Der erste Faktor ist der Spotpreis der Waren, modelliert nach einer geometrischen Brown'schen Bewegung mit zeitabhängigen Volatilitäten. Der zweite ist die aktuelle Verfügbarkeitsrendite[III], modelliert nach dem erweiterten Cox-Ingersoll-Ross (CIR) Prozess durch Hinzufügen einer zeitabhängigen Funktion zum Drift-Term des Prozesses, welche die saisonalen Änderungen der Warenpreise beschreibt. Die zeitabhängigen Volatilitäten des Spotpreises und der Verfügbarkeitsrendite sind proportional zur Quadratwurzel der aktuellen Verfügbarkeitsrendite. Unser Modell beschäftigt sich mit zwei relevanten Sachverhalten, und zwar dem Zusammenhang zwischen Volatilitäten des Preises und der Verfügbarkeitsrendite, der durch Lagerhaltungstheorie impliziert wird, und der Saisonalabhängigkeit von Warenpreisen und Volatilitäten der Verfügbarkeitsrendite. Wir geben hinreichende Bedingungen dafür an, dass die Volatilitäten strikt positiv bleiben. Lösungen in geschlossener Form für Futurespreise werden unter der Bedingung, arbitragefrei zu sein, hergeleitet. Die gefundenen Lösungen in geschlossener Form stimmen mit der Lagerhaltungstheorie überein: Futurespreis tendiert nämlich bei hinreichend größer Verfügbarkeitsrendite dazu, niedriger als der Spotpreis zu sein und auch umgekehrt. Außerdem führen diese Lösungen in geschlossener Form zu einer Formel für die Extraktion beider Faktoren, wenn zwei arbitragefreie Futurespreise mit verschiedenen Laufzeiten betrachtet werden können. Europäische Optionspreise der Futures werden dazu durch Fourier-Transformationen bestimmt. Wir schätzen die Parameter des Modells mit Hilfe von Daten der täglichen Futurespreise von zwei landwirtschaftlichen Erzeugnissen in Thailand, nämlich Reis und Naturgummi, ab. Diese Daten der Futurespreise stammen von „Agricultural Futures Exchange of Thailand“ (AFET) und beziehen sich auf die Zeitdauer vom August 2004 bis August 2006. Das Schätzverfahren basiert auf der Maximum-Likelihood-Methode. Die empirischen Ergebnisse stimmen mit der Lagerhaltungstheorie überein, und wir die thailändischen Regeln für Preisintervention berücksichtigen. Für einige Futureskontrakte dieser zwei Waren berechnen wir mit Hilfe der abgeschätzten Parameter Preisunterschiede und Korrelationen zwischen den betrachteten Futurespreisen und den anhand unseres Modells vorhergesagten Futurespreisen. Den Ergebnissen zufolge sind die betrachteten und die vorhergesagten Futurespreise nicht signifikant unterschiedlich und stark positiv korreliert. Schließlich analysieren wir die Folgerungen unseres Modells für Entscheidungen zur Kapital-Budgetierung durch Untersuchung der Situationen namens Backwardation und Contango in AFET. Wir haben für Futureskontrakte langer Laufzeit gefunden, dass der Futuresmarkt von Reis die Backwardation zeigt, während der von Naturgummi das Cotango zeigt.

[III] Verfügbarkeitsrendite heißt auf Englisch "convenience yield".

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Sanae Rujivan
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Contents

Life and Price are "Random Walks",

but Mankind tries to control them.

Sanae

Introduction

In the last three decades, the development of commodity futures markets has focused on the necessity of developing new models of commodity prices in order to price commodity futures and other commodity derivatives. In the current literature and practice, stochastic models of commodity prices play a crucial role because these models treat commodity spot prices as "a random walk" and provide closed-form solutions to evaluate futures and some other commodity derivatives under economic constraints. This in turn allows for a relatively easy calibration and computational implementation of these models. Basically, this approach considers the commodity price and the convenience yield as two different stochastic processes with constant correlation. This class of models was first proposed by Brennan-Schwartz (1985) [B-03] where the commodity price follows a Geometric Brownian Motion (GBM) and the convenience yield is described in the same way as a dividend yield. Nevertheless, this specification is inappropriate because it does not take into account the mean-reversion property of the commodity prices and ignores the inventory-dependence property of the convenience yields.

Gibson-Schwartz (1990) [G-03] introduced a two-factor model with a constant volatility in which the commodity price and the convenience yield follow a joint stochastic process with constant correlation. Specifically, the commodity spot price follows a GBM and the instantaneous convenience yield is taken as a second state-variable following a mean reverting stochastic process of the Ornstein-Uhlenbeck (OU) type (or the Vasicek model). The two state variables are only linked through a coefficient of correlation. The OU process relies on the hypothesis that there is a regeneration property of inventories, namely, there is a level of stocks which satisfies the needs of industry under normal conditions. The behavior of the operators in the physical market guarantees the existence of this normal level. When the convenience yield is low, the stocks are abundant and the operators sustain a high storage cost compared with the benefits related to holding the commodity. Therefore, if the holders are rational, they try to reduce these surplus stocks. Conversely, when the stocks are rare the operators tend to reconstitute them. Schwartz (1997) [S-01] introduced variation of this model in which the convenience yield is mean reverting and intervenes in the commodity price dynamics. Schwartz (1997) [S-01] -Model 3, Miltersen-Schwartz (1998) [M-01] and Hilliard-Reis (1998) [H-03] included a third stochastic factor to the model to account for stochastic interest rates. However, the inclusion of stochastic interest rates in the commodity price models does not have a significant impact in the pricing of commodity options and futures in practice. Accordingly, interest rate can be assumed deterministic. Nielson-Schwartz (2004) [N-02] extended the literature on commodity pricing by incorporating a link between the spread of forward prices and spot price volatility as proposed in Fama-French (1988) [F-01] and Ng-Pirrong (1994) [N-01]. The model in Nielson-Schwartz (2004) [N-02] allows the return volatilities to depend on the level of convenience yield as suggested by the theory of storage.

Besides the mean-reversion property of commodity prices, the other main empirical characteristic that makes commodities strikingly different from stocks, bonds, and other conventional financial assets, is seasonality in prices. Many commodities, such as agricultural commodities or natural gas, exhibit seasonality in prices, due to harvest cycles in the former case and changing consumptions as a result of weather patterns in the latter case. In term structure model of commodity prices, some research has been conducted on the seasonality of commodity prices. Sørensen (2002) [S-02] began with a model describing the dynamics of the (log-) commodity spot price as the sum of a deterministic seasonal component, a non-stationary state-variable, and a stationary state-variable. The deterministic term in seasonal component is modeled by a parameterized linear combination of trigonometric functions with seasonal frequencies. The non-stationary state-variable is modeled by the logarithm of a geometric diffusion process, well-known from the standard Black-Scholes setting, and is included in order to describe permanent price changes due to for example technology improvements, permanent changes in demand/taste, and general price increases due to common inflation. The stationary state-variable is modeled by an OU process which is included in order to capture the mean-reverting feature of commodity prices. The same type of formalization was introduced by Richter-Sørensen (2004) [R-01] and Geman-Nguyen (2005) [G-02]. The models take the commodity spot price, the instantaneous convenience yield, and the volatility of the convenience yield and the volatility of commodity price as separated state-variables. Nevertheless, those three models do not allow the return volatilities to depend on the level of convenience yield as suggested by the theory of storage.

In this research, we develop a two-factor model of commodity prices which is an extension of the model proposed by Nielsen-Schwartz (2004) [N-02] in the following form.

The Model

$$\begin{aligned} dS_t &= (r - \delta_t + \lambda_S(\beta_1\delta_t + \beta_2))\,S_t dt + \sqrt{\beta_1\delta_t + \beta_2}\; S_t\, dW_t^{(1)} \\ d\delta_t &= (\alpha_T(t) - \kappa\delta_t + \lambda_\delta(\beta_1\delta_t + \beta_2))dt + \sigma_\delta\sqrt{\beta_1\delta_t + \beta_2}\; dW_t^{(2)}. \end{aligned} \tag{M}$$

The first factor is the commodity spot price S_t which follows a GBM with a time-varying volatility, which is proportional to the square root of the instantaneous convenience yield. The second factor is the instantaneous convenience yield δ_t which follows an extended Cox-Ingersoll-Ross (CIR) process where a deterministic seasonal function $\alpha_T(t)$ is added in the drift term of the process. The correlation between the two processes is assumed constant. The direct proportionality of the commodity price volatilities and the convenience yield volatilities to the square root of the convenience yields reflects the effect of supply, demand, inventory, and seasonality in the commodity prices and the convenience yield volatilities as previously suggested in the literature. Additionally, in order to price futures and options contracts of the commodity, we assume that the following assumptions hold.

The No-Arbitrage Assumptions

(1) The market is *arbitrage-free*, that is, for any portfolio $\varphi = (\varphi_t)$,

$$V_\varphi(0) = 0 \text{ and } V_\varphi(T) \geq 0,\ \mathbb{P}\text{- a.s. for all time } T > 0 \text{ imply } V_\varphi(T) = 0,\ \mathbb{P}\text{- a.s.,}$$

where $V_\varphi(t) \equiv V_\varphi(t, S_t, \delta_t, \varphi_t)$ denotes the value of the portfolio φ at time t and $\mathbb{P}$ denotes an original probability measure. Namely, if a portfolio requires a null investment and is riskless (there is no possible loss at the time horizon T), then its terminal value at time T has to be zero.

(2) The market participants are subject to no transaction costs when they trade.

(3) The market participants are subject to no tax rate on all net trading profits.

(4) The market participants can borrow/lend money at the same risk free rate of interest.

Under the no-arbitrage assumptions, the fair-prices (or the no-arbitrage prices) of futures and options contracts can be determined under a so-called equivalent martingale measure (or the risk-neutral probability measure) $\mathbb{Q} \sim \mathbb{P}$.

The aim of this dissertation is threefold.

(I) For a futures market of a commodity, by assuming that the commodity spot prices and the instantaneous convenience yields follow model (M) under an equivalent martingale measure $\mathbb{Q}$, we want to derive no-arbitrage prices of the commodity futures and the European options written on the commodity futures.

(II) By assuming that the commodity spot prices and the instantaneous convenience yields cannot be observed in the market, we want to extract the commodity spot prices and the instantaneous convenience yields from the corresponding no-arbitrage futures prices.

(III) We want to calibrate model (M) by using empirical futures prices data. Namely, the model parameters have to be estimated by using observed futures prices data from the futures market. Using the estimated parameters, we want to demonstrate the practical applicability of model (M) by calculating price differences and correlations between the observed futures prices and their corresponding no-arbitrage (predicted) futures prices obtained from model (M). Finally, we want to analyze the implications of model (M) for capital budgeting decisions by investigating the situations known as backwardation and contango in the futures market.

In Chapter 1, we solve the problems imposed in (I). We obtain closed-form solutions for the futures prices. Numerical solutions for the futures options prices are determined by using a method of Fourier transforms. The closed-form solutions for the futures prices are consistent with the theory of storage such that futures prices tend to be lower than spot prices when convenience yields are sufficiently high and vice versa. Moreover, the closed-form solutions lead to the determination of the two state-variables as desired in (II). We achieve (III) by employing a method of maximum likelihood. These works are done in Chapter 2 and Chapter 3. In Chapter 2, we construct a sequence of closed-form approximations of the transition density of the logarithm futures prices process, and hence, the log-likelihood function of log-futures prices data and prove its convergence in probability to the true log-likelihood function. This convergence implies that the limit of the sequence of approximate maximum likelihood estimators is close to the true maximum likelihood estimators which can be inferred to the true-parameters describing the dynamics of the process. In Chapter 3, we apply model (M) to two agricultural commodities in Thailand, rice and natural rubber. We use the daily futures prices data of the two commodities obtained from the Agricultural Futures Exchange of Thailand (AFET) in two time periods in August 2004 to August 2006.

The estimation results show a clear seasonal pattern in both price and convenience yield volatilities of the two commodities. In addition, the numerical results indicate that the convenience yields tend to be high when the inventory/supply is low, and vice versa. However, there is an impact from the Thai price intervention scheme on the domestic rice prices which can be noticed from the daily extracted rice price volatilities such that they are less variable than the daily extracted rubber price volatilities. Price differences and correlations between the observed and the predicted futures prices are computed for several futures contracts of rice and natural rubber. The results obtained show that the price differences are insignificantly different from zero and the correlations are highly positive. This implies that our model is applicable for the two commodities prices. Furthermore, we observe that, for each selected futures contract, the price differences on the days close to its maturity date are hardly realized by the market participants. In the economic point of view, these results can be explained by the equilibrium in the futures market, namely, the observed futures prices approach their corresponding no-arbitrage futures prices when the futures market is close to the equilibrium. Finally, we investigate the situations known as backwardation and contango in AFET by observing on the forward surfaces for the two commodities in the sample periods. We have found that, for long maturity futures contracts, the futures market of rice exhibited backwardation, while the futures market of natural rubber exhibited contango. These results can be explained as follows. In the long run futures of the two commodities prices, the market has expected a decrease in rice prices, but an increase in natural rubber prices.

The remaining of this dissertation is organized as follows.

In Chapter 1, we briefly review the theory of storage as a motivation to develop a model of commodity prices. Then model (M) is presented in a concrete mathematical way. Sufficient conditions on the convenience yield process are given to ensure that the volatilities are always positive. Milstein scheme is used to simulate sample paths of the two state variables and then Monte Carlo method is employed to evaluate approximate no-arbitrage futures prices. The closed-form no-arbitrage futures prices are derived in both cases, the convenience yields are assumed deterministic and stochastic. Using two no-arbitrage futures prices having different maturities, we derive extraction formulas for the two state-variables. Applying the Itô formula, we write down the dynamics of logarithmic futures prices which are used as the underlying process in estimation of the parameters based on the maximum likelihood approach. The numerical solutions for European futures option prices are derived in the last subsection of the chapter.

In Chapter 2, we solve the forward Kolmogorov equation to obtain the forward transition density of the log-futures prices process. Since the obtained forward transition density contains integral terms of a discretely observed function, we derive a closed-form approximation of the forward transition density by using observed futures prices data and investigate the error estimate. The remaining of the chapter is devoted to a construction of the approximate log-likelihood function of logarithmic futures prices data and the poof of its convergence in probability sense to the true log-likelihood function as previously mentioned.

In Chapter 3, we calibrate model (M) using the daily futures prices data of rice and natural rubber obtained from AFET. We start the chapter by informing the backgrounds in rice and rubber productions and prices in Thailand. Next, we explain more precisely about the futures prices data of rice and rubber in terms of contract specifications and the sample time periods. Using the results obtained from Chapter 1 and Chapter 2, we specify the parameters set and the constraints and then we estimate the model parameters based on the maximum likelihood approach. The optimization problems arising with the use of the estimation method are solved using a heuristic algorithm known as Differential Evolution (DE) provided in *Mathematica*. The estimation results are reported with discussions focusing on the implications for the prices of rice and natural rubber in Thailand. Using the estimated parameters, we compute the corresponding no-arbitrage (predicted) futures prices. Then we introduce the measurements for price differences and correlations between the observed futures prices and the predicted futures prices. We report the price differences and the correlations of the two commodities via two series of graphs and a discussion about the results obtained is provided thereafter. In the last section of the chapter, we analyze the implications of model (M) for capital budgeting decisions. We display the forward surfaces for the two commodities obtained from model (M) in the sample periods and then a discussion about the situations known as backwardation and contango observed on the forward surfaces is provided therein.

We finally conclude this dissertation with summaries and an outlook of interesting future developments in modeling of commodity prices.

Chapter 1

Stochastic Modeling for Commodity Prices and Valuation of Commodity Derivatives under Stochastic Convenience Yields and Seasonality

This chapter starts by introducing the conceptual ideas in the theory of storage relating to the stochastic behavior of commodity prices. Then model (M) introduced in Introduction is described in a concrete mathematical way. Sufficient conditions on the convenience yields process are given for the inaccessibility to nonpositive values of the volatilities. Milstein scheme is run for simulating sample paths of the two state variables and Monte Carlo technique is employed to evaluate approximate no-arbitrage futures prices. Furthermore, we derive closed-form solutions for no-arbitrage futures prices in both cases: the convenience yields are assumed deterministic and stochastic. Subsequently, we derive extraction formulas for the two state-variables under the assumption that two no-arbitrage futures prices having different maturities can be observed. Using the Itô formula, we write down the dynamics of logarithmic futures prices which will be used in Chapter 2 as the underlying process in estimation of the model parameters based on a maximum likelihood approach. The remaining of this chapter is devoted to the derivation of the numerical solutions for European futures options prices.

1.1 Theory of Storage

The aim of this section is to briefly review the theory of storage to be a motivation for developing a stochastic model for commodity prices in the next section. The important term "convenience yield" arising from inventories of storable commodities, and the stochastic behaviors of convenience yields are explained and investigated, respectively.

1.1.1 Inventories and Convenience Yields

Consider a competitive commodity market subject to stochastic fluctuations in production and/or consumption. Market participants (producers, consumers, and possibly the third party) will hold inventories. These inventories serve a number of functions. Producers hold them to reduce costs of adjusting production over time, and also to reduce marketing costs by facilitating production and delivery scheduling and avoiding stock outs. Industrial consumers also hold inventories, and for the same reasons – to reduce adjustment costs and facilitate production (i.e., when the commodity is used as a production input), and to avoid stock outs. Pindyck (2001) [P-01] explained the function of inventory in a competitive commodity market that it acts as a "lubricant" for both producers and industrial consumers to mitigate the impacts of stochastic fluctuations in production and/or consumption.

In the cash market, purchases and sales of the commodity for immediate delivery occur at a price that we will refer to as the "spot price". Because inventory holding can change, the spot price does not equate production and consumption. Pindyck (2001) [P-01] characterized the cash market as a relationship between the spot price and "net demand", i.e., the difference between production and consumption. Let N_t denote the inventory level at time *t*. The change in inventory level at time *t*, denoted by ΔN_t, is given by

$$\Delta N_t = \mathbf{S}(S_t; z_t^{\mathbf{S}}, \varepsilon_t^{\mathbf{S}}) - \mathbf{D}(S_t; z_t^{\mathbf{D}}, \varepsilon_t^{\mathbf{D}}) \tag{1.1.1}$$

where S_t is the spot price at time *t*, $\mathbf{S}(S_t; z^{\mathbf{S}}, \varepsilon^{\mathbf{S}})$ is a supply function, $\mathbf{D}(S_t; z^{\mathbf{D}}, \varepsilon^{\mathbf{D}})$ is a demand function, $z^{\mathbf{S}}$ is a vector of supply-shifting variables, $z^{\mathbf{D}}$ is a vector of demand-shifting, $\varepsilon^{\mathbf{S}}$ and $\varepsilon^{\mathbf{D}}$ are random shocks such as from unpredictable changes in tastes and/or technologies.

Equation (1.1.1) indicates that the cash market is in equilibrium when net demand equals net supply and then we obtain the following inverse net demand function:

$$S_t = S_t(\Delta N_t; z_t^{\mathbf{D}}, z_t^{\mathbf{S}}, \varepsilon_t^{\mathbf{D}}, \varepsilon_t^{\mathbf{S}}). \tag{1.1.2}$$

Pindyck (2001) [P-01] concluded that because $\partial \mathbf{S} / \partial S_t > 0$ and $\partial \mathbf{D} / \partial S_t < 0$, the inverse net demand is upward sloping in ΔN_t, i.e., a higher price corresponds to a larger **S** and smaller **D**, and thus a larger ΔN_t. He also pointed out that an increase in price volatility implies an increase in the demand for inventory. In other words, price volatility has a negative relationship with inventory level, i.e., an increase in inventory level can reduce price volatility. Other things equal, market participants will want to hold greater inventories in order to buffer these fluctuations in production and consumption. On the relationship between the demand for inventory and commodity price, he concluded that one should be willing to pay more to store a higher-priced good than a lower-priced one. When inventory holdings can change, production in any period need not equal consumption. As a result, the market-clearing price is determined not only by current production and consumption, but also by changes in inventory holdings.

The calculation of profitability for holding inventories of the commodity spot rests on the determination of the influence of inventories, production and consumption on the expected spot price. In the commodity literature, supply of storage theory describes this relationship. An excellent summary of these concepts appears in Cootner (1967) [C-01].

The direct costs of holding inventory include warehouse rental and insurance. These costs are thought to be relatively constant over a wide range of inventory levels. Holding inventory ties up capital. An implicit interest charge is usually including in the direct costs of carrying inventory. The difference between futures price and spot price is termed the *basis*. When the direct costs of carrying inventory are netted out of the basis, there is an empirically substantiated residual component termed *the marginal convenience yield*. This relationship is depicted in Figure 1.1 (the "supply of storage" curve).

The classic explanation of the marginal convenience yield phenomena is that convenient inventories provide benefits to the holders of inventory by reducing stock out costs. Also, convenient inventories reduce the chance of turning away good customers (or increase the chance of adding new customers) when inventories are scarce. Thus this theory holds that consumers will pay inventory holders for reducing their costs to locate supplies in times of scarcity. While there has been no completely satisfactory explanation of the supply of storage phenomena, the empirical effect exists for many commodities (e.g., Brennan (1958) [B-04], Working (1949) [W-01]).

Net Convenience yields on a commodity can be thought of in the same way as dividend yields on a common stock. Net convenience yields can be separated into gross convenience yields and costs of carry. Gross convenience yield is the value of all the advantages of possessing the commodity, whereas the cost of carry is the cost of the disadvantages. The net convenience yield is the result of subtracting the cost of carry from the gross convenience yield and it can in many cases be negative. Normally the convenience yield is quoted as a continuously compounded yield (as a continuous compounded interest rate). Under the no-arbitrage assumptions, it is well known that the spot-forward relationship holds for any storable commodity, i.e.,

$$F_t^T = S_t e^{(r-\delta)(T-t)} = S_t e^{(r-(c^+-c^-))(T-t)}, \qquad T > t, \tag{1.1.3}$$

where F_t^T is the forward (or futures) price[1] of the commodity on day t maturity at date T, S_t is the commodity spot price on day t, r is the continuously compounded interest rate, δ is the net convenience yield, c^+ is the gross convenience yield, c^- is the cost-of-carry yield. These constants are taken over the period from day t to day T.

[1] Under non-stochastic interest rates, forward and futures prices for the same underlying and maturity are equal (assuming no credit risk in the forward contract transaction).

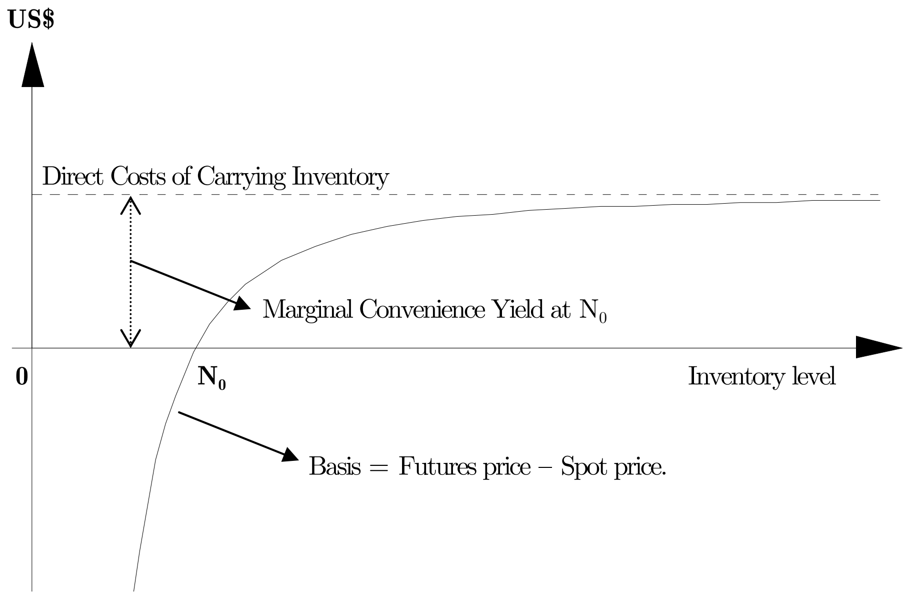


Figure 1.1: The "Supply of Storage" curve

We summarize the important implications of the theory of storage as follows.

S1: The volatility of a commodity spot price tends to be inversely related to the level of global stock. In the case of stock outs, spot prices change dramatically in response to supply and demand.

S2: The price of a commodity, its volatility, and the convenience yield, are positively correlated since they are negative related to the inventory level. This feature is called the "*inverse leverage effect*", namely, the positive relationship between commodity prices and their volatility.

S3: Convenience yields fluctuate considerably overtime. Some of these fluctuations are predictable, in that they correspond to "*seasonal variation*" in the demand for storage. Much of the variation in convenience yield, however, is unpredictable, and corresponds to unpredictable temporary fluctuations in demand or supply in the cash market.

S4: Inventory and demand conditions affect the variances of commodity spot prices and the convenience yields. In addition, under the hypothesis that spot return dynamics are strongly related to variations in fundamental supply and demand conditions, the spot return volatility is increasing in the level of convenience yield (see Fama-French (1988) [F-01] and Ng-Pirrong (1994) [N-01]).

1.1.2 Mean-Reversion and Seasonality in Commodity Prices

Mean-Reversion is one of the main properties that has been systematically incorporated in the recent literature on commodity price modeling. Commodity prices neither grow nor decline on average over time, but they fluctuate around their long-run mean. In other words, they tend to mean-revert to a level which may be viewed as the marginal cost of production. This has been evidenced a number of times in literature (see, for instance, Pindyck (2001) [P-01] for energy commodities and Geman-Nguyen (2005) [G-02] for the case of agricultural commodities).

In a nondeterministic setting the resemblance with interest rates and dividends is preserved. Stochastic models of commodity price behavior typically include both stochastic process for the commodity prices and a separate stochastic process for the convenience yields (Gibson-Schwartz (1990) [G-03], Schwartz (1997) [S-01], Richter-Sørensen(2004) [R-01], and Nielsen- Schwartz (2004) [N-02]). Often, a stochastic process for the convenience yields is modeled in the same way as a stochastic process for interest rates such as the Vasicek model or the Cox-Ingersoll-Ross (CIR) model. These models reflect the mean-reversion behavior of convenience yields as previously described in Introduction.

Besides the mean-reversion behavior of commodity prices, the other main empirical characteristic that makes commodities strikingly different from stocks, bonds, and other conventional financial assets, is seasonality in prices (e.g., the discussion in Routledge-Seppi-Spatt (2000) [R-02]). Many commodities, such as agricultural commodities or natural gas, exhibit seasonality in prices, due to harvest cycles in the former case and changing consumption as a result of weather patterns in the latter case. In term structure model of commodity prices, some research has been conducted on the seasonality of commodity prices. Sørensen (2002) [S-02] began with a model describing the dynamics of the (log-) commodity spot price as the sum of a deterministic seasonal component, a non-stationary state-variable, and a stationary state-variable. The deterministic term in seasonal component is modeled by a parameterized linear combination of trigonometric functions with seasonal frequencies. The same type of formalization was introduced by Richter-Sørensen (2004) [R-01] and Geman-Nguyen (2005) [G-02]. The models take the commodity spot price, the instantaneous convenience yield, and the volatility of the convenience yield and the volatility of commodity price as separated state-variables. Nevertheless, those three models do not allow the return volatilities to depend on the level of convenience yield as suggested by the theory of storage.

1.2 Stochastic Modeling for Commodity Prices

Stochastic modeling for commodity prices plays an important role in pricing commodity derivatives such as futures contracts and options, under the no-arbitrage assumptions. Early studies in this area typically assumed that the commodity spot prices and the instantaneous convenience yields are random and they are followed a joining stochastic process with constant correlation. The excellent literature review on stochastic models of commodity spot prices can be found in Lautier (2003) [L-01]. Using those models, one can derive fair-prices of futures or options either in closed-form solutions or numerical solutions. In this section, we present a stochastic model of commodity spot prices based on the theory of storage and seasonality as described in the previous section. Sufficient conditions for the convenience yield process are proposed for having volatilities of the commodity prices to be meaningful. The Mote Carlo Simulation is employed to evaluate the approximate futures prices. Moreover, under deterministic convenience yields, the no-arbitrage futures prices are derived.

1.2.1 The Model

The model of commodity spot prices developed in this research is an extension of the model proposed by Nielsen-Schwartz (2004) [N-02]. By incorporating seasonality into the model, the commodity spot prices process $(S_t)_{t\in[0,T]}$ and the instantaneous convenience yields process $(\delta_t)_{t\in[0,T]}$ under an equivalent martingale measure $\mathbb{Q}$ satisfy the following stochastic differential equations (SDEs):

$$\left.\begin{aligned} dS_t &= (r - \delta_t + \lambda_S(\beta_1\delta_t + \beta_2))\,S_t dt + \sqrt{\beta_1\delta_t + \beta_2}\; S_t\, dW_t^{(1)} \\ d\delta_t &= (\alpha_T(t) - \kappa\delta_t + \lambda_\delta(\beta_1\delta_t + \beta_2))dt + \sigma_\delta\sqrt{\beta_1\delta_t + \beta_2}\; dW_t^{(2)} \end{aligned}\right\}^{2} \tag{1.2.1}$$

with an initial condition (S_0, δ_0), where $T > 0$. In the model, r is the risk free interest rate, $\beta_i > 0, i = 1,2,$ are parameters measuring of the impact of convenience yields on the volatilities of the commodity spot prices. The deterministic seasonal function $\alpha_T(t)$ is of the following form:

$$\alpha_T(t) = \alpha_0 + f_\alpha(T-t)\left(\sum_{k=1}^{K^\alpha}\left(\alpha_k^{(1)}\cos(2\pi kt) + \alpha_k^{(2)}\sin(2\pi kt)\right)\right), \tag{1.2.2}$$

where K^α determines the number of terms in the summation and $\alpha_0, \alpha_k^{(1)}, \alpha_k^{(2)}$, $k = 1$, $\ldots, K^\alpha$, are constant parameters. The non-specific function $f_\alpha(t)$ is a positive function determining the magnitude of the periodic terms on the RHS of Equation (1.2.2) at time t.

[2] See the derivation of the model (1.2.1) in Appendix A.

It should be noted that the form of $\alpha_T(t)$ is a flexible and natural choice of modeling the seasonal aspects of commodity price behavior in continuous time which is also applied in Richter-Sorensen (2004) [R-01] with the case that $f_\alpha \equiv 1$. In this research, we choose

$$f_\alpha(t) = \frac{\sqrt{p} + p_2 + (\sqrt{p} - p_2)\, e^{\sqrt{p}\, t}}{e^{\sqrt{p} t}}, \quad |p_2| < \sqrt{p}, \tag{1.2.3}$$

for $0 \le t \le T$, where the constants p and p_2 are given in Proposition 5. With this choice of f_α, $\alpha_T(t)$ still behaves as the case $f_\alpha \equiv 1$ on $[0, T]$ which can be used to describe the seasonal variation in the convenience yield process[3]. Moreover, we are able to derive closed-form solutions for futures prices as expressed in Proposition 5.

The parameter $\kappa > 0$ is the magnitude of the speed of the convenience yields measuring the degree of reversion to the deterministic seasonal pattern in the convenience yield. The parameter $\sigma_\delta > 0$ is the magnitude of the impact of convenience yields on volatility of themselves. We let $W \equiv (W_t^{(1)}, W_t^{(2)})_{t\in[0,T]}$ denote a two-dimensional Brownian motion under the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with a filtration $(\mathcal{F}_t)_{t\in[0,T]}$. The two Brownian motions are correlated with a constant $\rho \in (-1,1)$, i.e.,

$$dW_t^{(1)} dW_t^{(2)} = \rho\, dt \ \text{ for all } t \in [0, T]. \tag{1.2.4}$$

In our setting, we have assumed that there are no assets that are clearly instantaneously perfectly correlated with the state variables S_t and δ_t. In other words, the values of S_t and δ_t cannot be observed in our setting. Thus, it does not seem possible to construct a hedge portfolio that eliminates all the risk. This causes the risk premium of spot price and the convenience yield risk arise (as proposed by Hull-White (1987) [H-04]) and they are added in the drift terms of the state variables S_t and δ_t, respectively. In our model, we assume that the risk premium of the commodity spot prices is proportional to the variance of commodity spot prices and the convenience yield risk is proportional to the variance of instantaneous convenience yield (as suggested by Cox-Ingersoll-Ross (1985) [C-02]). We let λ_S and λ_δ denote the constants of the proportionalities (see Appendix A).

The model (1.2.1) has two stochastic factors. The first factor is the commodity spot price which follows a Geometric Brownian Motion (GBM) with a time-varying volatility. The second factor is the instantaneous convenience yield which follows an extended CIR process with adding the deterministic seasonal function $\alpha_T(t)$ in the drift term of the process.

[3] At the end of Appendix D, we provide some properties of f_α.

The time-varying volatilities of the commodity spot prices and the instantaneous convenience yields are proportional to the square root of the instantaneous convenience yields. The correlation between the commodity spot price process and the instantaneous convenience yield process ρ is assumed constant. The direct proportionality of the commodity spot price volatilities and the instantaneous convenience yield volatilities to the square root of the instantaneous convenience yields reflect the effect of supply, demand, inventory, and seasonality in the commodity prices and the convenience yield volatilities as suggested by the theory of storage.

In this research, we assume that the risk free interest rate r is known. The $8+2K^{\alpha}$ unknown parameters contained in the vector θ defined by

$$\theta := (\beta_1, \beta_2, \kappa, \sigma_\delta, \lambda_S, \lambda_\delta, \rho, \alpha_0, \alpha_1^{(k)}, \alpha_2^{(k)}),\ \ k = 1, 2, ..., K^{\alpha},$$

will be estimated based on a maximum likelihood approach. Since we cannot observe S_t and δ_t from the market under this setting, only futures prices are available in the market. Hence, we use the logarithmic futures prices process as the underlying process instead of S_t and δ_t to construct a closed-form approximation of the true likelihood function of the logarithmic futures prices data. The closed-from approximation will be used as an objective function for constructing approximate maximum likelihood estimators of the unknown parameters. These works will be done in Chapter 2 and Chapter 3.

1.2.2 Sufficient Conditions for the Convenience Yields Process

In this research, we allow δ_t can be either negative or positive or zero because it is the difference between the gross convenience yield c_t^+ and the cost-of-carry yield c_t^-, i.e.,

$$\delta_t = c_t^+ - c_t^-, \tag{1.2.5}$$

for all $t \in [0, T]$. However, having the volatilities of S_t and δ_t to be meaningful, the following condition must be satisfied:

$$\delta_t > -\tfrac{\beta_2}{\beta 1}, \quad \mathbb{Q}\text{- a.s. for all } t \in [0, T], \tag{1.2.6}$$

which is equivalent to the condition

$$\hat{\delta}_t := \beta_1 \delta_t + \beta_2 > 0,\ \ \mathbb{Q}\text{- a.s. for all } t \in [0, T]. \tag{1.2.7}$$

The condition (1.2.6) tells that the instantaneous convenience yields must be bounded from below over the time period of consideration. Suppose that $c_t^+ \equiv 0$, the positive ratio β_2 / β_1 interprets as the maximum cost rate of carrying yields.

Since the dynamics of $\hat{\delta}_t$ is an extension of the CIR model of the form:

$$d\hat{\delta}_t = (\upsilon_T(t) - (\kappa - \lambda_\delta \beta_1)\hat{\delta}_t)dt + \sigma_\delta \beta_1 \sqrt{\hat{\delta}_t}\, dW_t^{(2)}, \tag{1.2.8}$$

where

$$\upsilon_T(t) := \beta_1 \alpha_T(t) + \kappa \beta_2. \tag{1.2.9}$$

Therefore, some conditions on the parameters must be imposed to ensure that nonpositive values are inaccessible to the transformed process (1.2.8), namely, the volatilities of the commodity spot prices and the instantaneous convenience yields must be positive.

Proposition 1.

For given $\hat{\delta}_0 > 0$, a sufficient condition for the inaccessibility of $\hat{\delta}_t$ to nonpositive values is

$$\frac{\beta_1 \left(\alpha_0 - f_\alpha(T;\theta) \sum_{k=1}^{K^\alpha} \left| \alpha_k^{(1)} \right| + \left| \alpha_k^{(2)} \right| \right) + \kappa \beta_2}{\sigma_\delta^2 \beta_1^2} \geq \frac{1}{2}. \tag{1.2.10}$$

Moreover, under this condition, for given $(S_0, \delta_0) \in D := (0,\infty) \times (\frac{-\beta_2}{\beta_1}, \infty)$, there exists a unique strong solution $X \equiv (S_t, \delta_t)_{t \in [0,T]}$ of the SDEs (1.2.1) with $X_0 = (S_0, \delta_0)$ and X never explodes or leaves D before T, $\mathbb{Q}$ - a.s..

We proved Proposition 1 in Appendix B by applying a Comparison Theorem for Solution of Stochastic Differential Equations proposed by Zhiyaun (1984) [Z-01].

1.2.3 No-Arbitrage Futures Prices and Monte Carlo Simulation

A fundamental implication of asset pricing theory is that, under the no-arbitrage assumptions, the fair-price of a derivative security (futures or option contract) at a current time can be represented by the expected value of its discounted payoff function at the maturity date under a risk-neutral probability measure. In fact, valuing derivatives reduces to computing the expectation with respect to the probability measure. In terms of pricing futures contracts, the following theorem is necessary.

Theorem 1.2.1.

Under the no-arbitrage assumptions in a futures market, the no-arbitrage futures price on day t with maturity date T, denoted by F_t^T, must satisfy

$$F_t^T = E_{\mathbb{Q}}\left[S_T \mid \mathcal{F}_t \right], \tag{1.2.11}$$

where the expectation is taken under a risk-neutral probability measure $\mathbb{Q}$ conditioned on the information $\mathcal{F}_t$.

Relation (1.2.11) tells that the no-arbitrage futures price today is an unbiased estimator of the spot price at the maturity date of the contract where we consider under the risk-neutral probability measure and the information available today.

To apply the Mote Carlo approach for evaluating the approximate no-arbitrage futures prices, there are three steps as summarized by the following:

(M1) Simulate a sample path of the commodity spot prices by using the model (1.2.1) on the time interval $[t, T]$ to obtain $\tilde{S}^{(1)}(T)$ for the first simulation.

(M2) Repeat the procedure (M1) to obtain $\tilde{S}^{(i)}(T), i = 2, \ldots, N,$ for a large integer N.

(M3) Construct an estimator $\tilde{S}_N(T)$ of $E_{\mathbb{Q}}[S_T]$ as follows:

$$\tilde{S}_N(T) := \frac{1}{N}\sum_{i=1}^{N}\tilde{S}^{(i)}(T).$$

Suppose that $\tilde{S}^{(i)}(T)$, $i = 1, 2, \ldots, N,$ are independent and identically distributed random sample with mean $E_{\mathbb{Q}}[S_T]$ and variance $\tilde{\sigma}_T^2 < \infty$. We define the difference between the estimator $\tilde{S}_N(T)$ and $E_{\mathbb{Q}}[S_T]$ (or the random estimate for the mean error) as follows:

$$\hat{\mu} := \tilde{S}_N(T) - E_{\mathbb{Q}}[S_T].$$

We can decompose the random estimate $\hat{\mu}$ into two parts: that is as $\hat{\mu} = \mu_{sys} + \mu_{stat}$, where $\mu_{sys} := E_{\mathbb{Q}}[\hat{\mu}]$ denotes the systematic error and μ_{stat} denotes the statistical error. The central limit theorem asserts that, as $N \to \infty$, μ_{stat} is asymptotically normal distributed with mean zero and

$$\mathrm{Var}_{\mathbb{Q}}[\mu_{stat}] = \mathrm{Var}_{\mathbb{Q}}[\hat{\mu}] \approx \frac{\tilde{\sigma}_T^2}{N}. \tag{1.2.12}$$

Expression (1.2.12) implies that the standard error $\hat{\mu}$ tends to zero with $\sqrt{N}$ convergence rate. Hence, in order to obtain sufficiently small confidence intervals it is important to begin with a small variance in the random variable $\tilde{S}_N(T)$. With a direct simulation method one tries to fix the variance of $\tilde{S}_N(T)$ to a value which close to that of the variance of S_T. However, this variance, which depends on the stochastic differential equations (1.2.1), may sometimes be extremely large. This leads to the problem of variance reduction which is beyond the scope of this research.

To simulate sample paths of commodity spot prices and instantaneous convenience yields, we transform the model (1.2.1) to a diffusion model driving on two independent Brownian motions $\tilde{W}^{(1)}$ and $\tilde{W}^{(2)}$. The transformed model is of the following form:

$$\left.\begin{aligned} dS_t &= b_1(t,S_t,\delta_t)\,dt + \sigma_{11}(t,S_t,\delta_t)\,d\tilde{W}_t^{(1)} + \sigma_{12}(t,S_t,\delta_t)\,d\tilde{W}_t^{(2)} \\ d\delta_t &= b_2(t,S_t,\delta_t,)\,dt + \sigma_{21}(t,S_t,\delta_t)\,d\tilde{W}_t^{(1)} + \sigma_{22}(S_t,\delta_t,t)\,d\tilde{W}_t^{(2)} \end{aligned}\right\} \tag{1.2.13}$$

where

$b_1(t,S_t,\delta_t) = (r - \delta_t + \lambda_S(\beta_1\delta_t + \beta_2))S_t$,

$b_2(t,S_t,\delta_t) = (\alpha_T(t) - \kappa\delta_t + \lambda_\delta(\beta_1\delta_t + \beta_2))$,

$\sigma_{11}(t,S_t,\delta_t) = \sqrt{\beta_1\delta_t + \beta_2}\,S_t$, $\sigma_{12}(t,S_t,\delta_t) = 0$,

$\sigma_{21}(t,S_t,\delta_t) = \sigma_\delta\,\rho\sqrt{\beta_1\delta_t + \beta_2}$, and $\sigma_{22}(t,S_t,\delta_t) = \sigma_\delta\sqrt{1-\rho^2}\sqrt{\beta_1\delta_t + \beta_2}$.

In order to have close pathwise approximations of the Itô processes in (1.2.13), we prefer the Milstein scheme to simulate sample paths of the processes. Under the regular conditions, the Milstein scheme converges with strong order 1.0 (see Kloeden-Platen (1999) [K-02]). First, we shall consider a time discretization $(t)_{\Delta t}$ with

$$t = t_0 < t_1 < \ldots < t_n < \ldots < t_M = T \tag{1.2.14}$$

on the time interval $[t,T]$ for some integer M, in which the equidistant case has step size

$$\Delta t = \frac{T-t}{M}. \tag{1.2.15}$$

The following recursive formulas derived by using the Milstein scheme are run for simulating the sample paths:

$$\begin{aligned} \tilde{S}_{n+1}^{(i)} &= \tilde{S}_n^{(i)} + b_1(t_n,\tilde{S}_n^{(i)},\tilde{\delta}_n^{(i)})\,\Delta t + \sum_{j=1}^{2}\sigma_{1j}(t_n,\tilde{S}_n^{(i)},\tilde{\delta}_n^{(i)})\,\Delta\tilde{W}_{t_n}^{(j,n)} + \sum_{j_1,j_2=1}^{2} L^{j_1}\sigma_{1j_2}(t_n,\tilde{S}_n^{(i)},\tilde{\delta}_n^{(i)})I_{(j_1,j_2)}^{(n)}, \\ \tilde{\delta}_{n+1}^{(i)} &= \tilde{\delta}_n^{(i)} + b_2(t_n,\tilde{S}_n^{(i)},\tilde{\delta}_n^{(i)})\,\Delta t + \sum_{j=1}^{2}\sigma_{2j}(t_n,\tilde{S}_n^{(i)},\tilde{\delta}_n^{(i)})\,\Delta\tilde{W}_{t_n}^{(j,n)} + \sum_{j_1,j_2=1}^{2} L^{j_1}\sigma_{2j_2}(t_n,\tilde{S}_n^{(i)},\tilde{\delta}_n^{(i)})I_{(j_1,j_2)}^{(n)}, \\ \tilde{S}_0^{(i)} &= S_t,\ \tilde{\delta}_0^{(i)} = \delta_t,\quad n = 0,1,\ldots,M-1, \text{ and } i = 1,2,\ldots,N, \end{aligned} \tag{1.2.16}$$

where $\Delta\tilde{W}_{t_n}^{(j,n)}, j = 1,2,$ are increments of the Brownian motion $\tilde{W}_{t_n}^{(j,n)}$ which are normal random variables with mean zero and variance Δt. The operators $L^j, j = 1,2,$ are of the following form:

$$L^j = \sigma_{1j}\frac{\partial}{\partial S} + \sigma_{2j}\frac{\partial}{\partial\delta}. \tag{1.2.17}$$

For $j_1 \neq j_2$ with $j_1, j_2 = 1, 2,$

$$I_{(j_1,j_2)}^{(n)} = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} d\tilde{W}_{s_2}^{(j_1,n)} d\tilde{W}_{s_1}^{(j_2,n)}, \tag{1.2.18}$$

are multiple Itô stochastic integrals, and for $j_1 = j_2$

$$I_{(j_1,j_2)}^{(n)} = \frac{1}{2}\left(\left(\Delta \tilde{W}_{t_n}^{(j_1,n)}\right)^2 - \Delta t\right). \tag{1.2.19}$$

We can approximate of the multiple Itô stochastic integral $I_{(j_1,j_2)}^{(n)}$ in Equation (1.2.18) by

$$\begin{aligned} I_{(j_1,j_2)}^{(n,p)} &= \Delta t\left(\frac{1}{2}\xi_{j_1}^{(n)}\xi_{j_2}^{(n)} + \sqrt{\rho_p}\left(\mu_{j_1,p}^{(n)}\xi_{j_2}^{(n)} - \mu_{j_2,p}^{(n)}\xi_{j_1}^{(n)}\right)\right) \\ &\quad + \frac{\Delta t}{2\pi}\sum_{r=1}^{p}\frac{1}{r}\left(\zeta_{j_1,r}^{(n)}\left(\sqrt{2}\xi_{j_2}^{(n)} + \eta_{j_2,r}^{(n)}\right) - \zeta_{j_2,r}^{(n)}\left(\sqrt{2}\xi_{j_1}^{(n)} + \eta_{j_1,r}^{(n)}\right)\right), \end{aligned} \tag{1.2.20}$$

where

$$\rho_p = \frac{1}{12} - \frac{1}{2\pi^2}\sum_{r=1}^{p}\frac{1}{r^2}, \tag{1.2.21}$$

and $\xi_j^{(n)}, \mu_{j,p}^{(n)}, \eta_{j,r}^{(n)},$ and $\zeta_{j,r}^{(n)}$ are independent standard normal random variables with

$$\xi_j^{(n)} = \frac{1}{\sqrt{\Delta t}}\Delta \tilde{W}_{t_n}^{(j,n)} \tag{1.2.22}$$

for $j = 1, 2,$ and for some $p = 1, 2, \ldots$ (see Kloeden-Platen (1999) [K-02]).

We consider three cases of parameters setting as tabulated in Table 1.1, such that for all cases, the parameters $r, \beta_1, \beta_2, \kappa, \rho,$ and α_0 are fixed. The number of terms in the summation of the seasonal function $K^{\alpha} = 2$ indicates that there is a possibility to have two local maxima and two local minima in the variation of the seasonal function on the time interval $[0, T], T = 1.$ In Case 2, we increase the volatility of the instantaneous convenience yields from Case 1 by 0.5. In Case 3, we neglect the risk premium of spot prices and the convenience yield risk and increase the magnitude of the seasonal terms. Figures 1.2 – 1.4 show the sample paths of commodity spot prices and instantaneous convenience yields obtained by simulations with the three cases of parameters setting. We do simulations over the time interval $[0,1]$ with the initial values $S_0 = 10, \delta_0 = 0.01,$ the number of time step $M = 300,$ and the number of simulations $N = 50.$

Table 1.1: Three Cases of Parameters Setting

Parameter	Case 1	Case 2	Case 3
r	0.10	0.10	0.10
β_1	0.05	0.05	0.05
β_2	0.01	0.01	0.01
κ	0.50	0.50	0.50
σ_δ	0.03	0.05	0.03
λ_S	0.05	0.05	0.00
λ_δ	0.01	0.01	0.00
ρ	0.50	0.50	0.50
K^α	2	2	2
α_0	0.05	0.05	0.05
$(\alpha_1^{(1)}, \alpha_2^{(1)})$	(0.05 , 0.05)	(0.05 , 0.05)	(0.10 , 0.10)
$(\alpha_1^{(2)}, \alpha_2^{(2)})$	(0.02 , 0.02)	(0.02 , 0.02)	(0.05 , 0.05)

Next, we evaluate $E_{\mathbb{Q}}[S_T \mid \mathcal{F}_t]$ based on Monte Carlo simulation. In this situation, it is not necessary to have close pathwise approximations of the Itô processes in (1.2.13). In simulating such the sample paths of the processes, it suffices to have a good approximation of the probability distribution of the random variable S_T rather than close approximations of the sample paths. Thus, the type of approximation required here is the weak convergence criterion and we prefer the Euler approximation which converges with weak order 1.0 (see Kloeden-Platen (1999) [K-02]).

By dropping the multiple Itô stochastic integral terms in Equation (1.2.16), the formulas become the Euler approximation. We set $N = 10,000,$ the distribution of the commodity spot prices at $t = 1,$ under Case 1 of parameter setting, is illustrated in Figure 1.5. The distribution is right-skewed with mean $\tilde{S}_N(1) = 10.6959 > S_0$. This result implies that, under this case, the market participants expect the higher spot price at $t = 1$ with a high probability. As shown in Figure 1.2, the instantaneous convenience yields are lower than the interest rate. This makes the futures price to be higher than the spot price at $t = 0$.

Overall, the Mote Carlo method proves to be flexible and easy to implement or modify. The method can deal with extremely complicated or high-dimensional problems. Moreover, the current advances in technology have reduced the computation time and have made the method to be more attractive. However, there are several disadvantages to this methodology; very complicated problems may require a very high number of simulations for an acceptable degree of accuracy and this may be rather time-consuming and expensive.

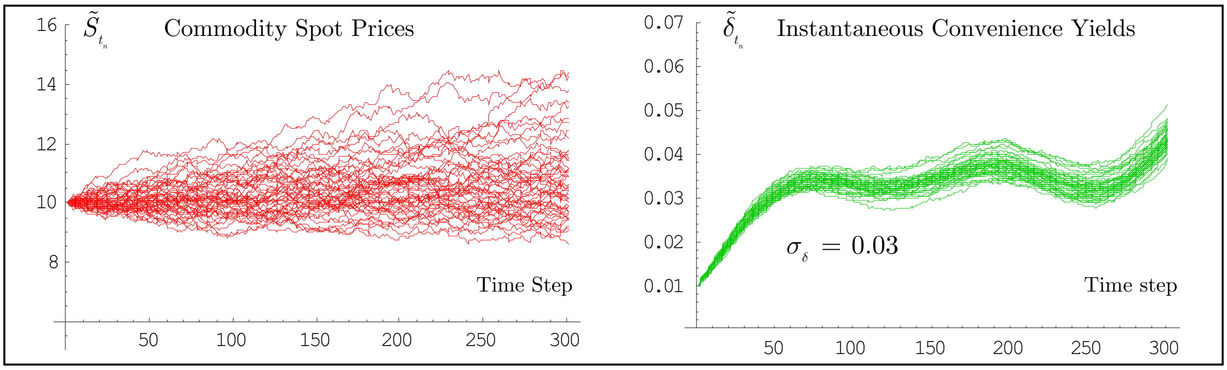


Figure 1.2: The sample paths Case 1

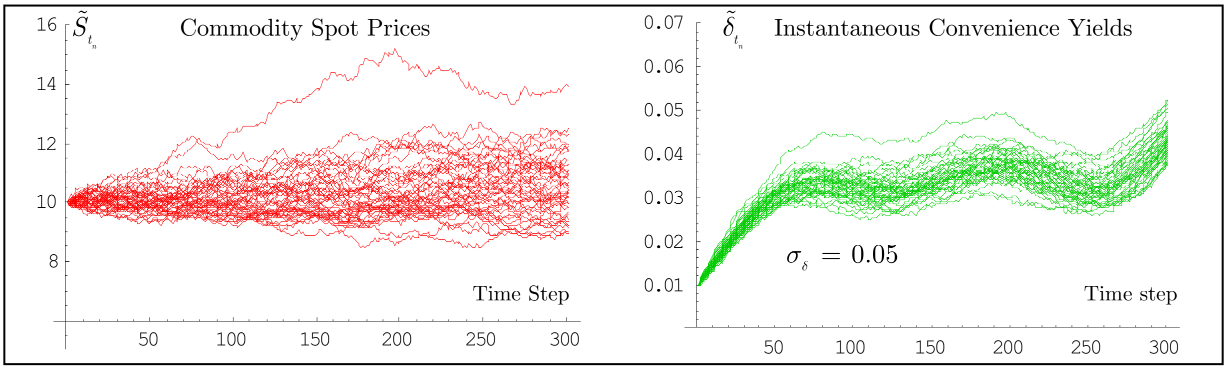


Figure 1.3: The sample paths Case 2

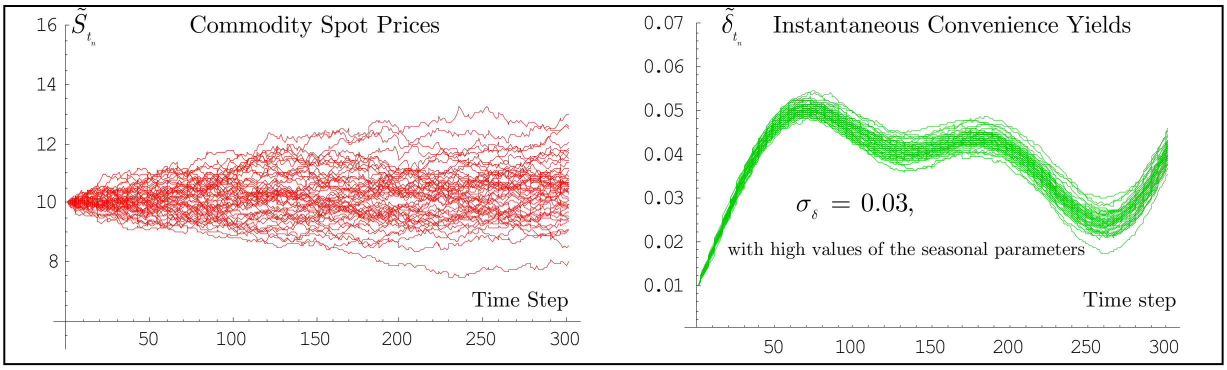


Figure 1.4: The sample paths Case 3

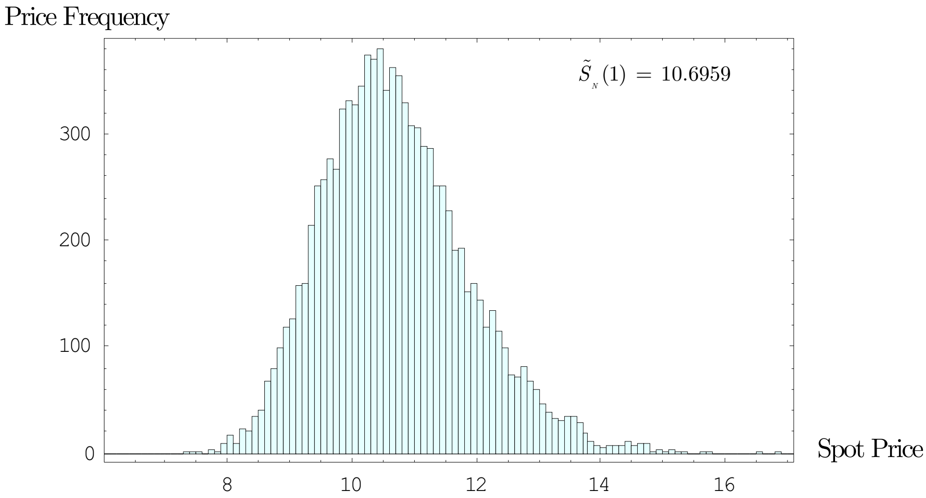


Figure 1.5: Distribution of the commodity spot prices under Case 1

1.2.4 Futures Prices under Deterministic Convenience Yields

Before deriving closed-form solutions for no-arbitrage futures prices under the assumption that instantaneous convenience yields are deterministic, we consider the following ordinary differential equations (ODEs):

$$\overline{S}'(t) = (r - \overline{\delta}(t) + \lambda_S(\beta_1\overline{\delta}(t) + \beta_2))\overline{S}(t), \tag{1.2.23}$$

$$\overline{\delta}'(t) = (\alpha_T(t) - \kappa\overline{\delta}(t) + \lambda_\delta(\beta_1\overline{\delta}(t) + \beta_2)), \tag{1.2.24}$$

for $t \in (0, T]$, where, under this consideration, $\overline{S}(t)$ and $\overline{\delta}(t)$ denote, respectively, the commodity spot price and the instantaneous convenience yield at time t.

Proposition 2.
For given initial conditions $\overline{S}(0) = \overline{S}_0$, $\overline{\delta}(0) = \overline{\delta}_0$, with the condition $\kappa - \lambda_\delta\beta_1 > 0$, the solutions to the ODEs (1.2.23) and (1.2.24) are

$$\overline{S}(t) = \overline{S}_0 e^{\left(r+\lambda_S\beta_2-\frac{(\alpha_0+\lambda_\delta\beta_2)(1-\lambda_S\beta_1)}{\kappa-\lambda_\delta\beta_1}\right)t-(1-\lambda_S\beta_1)\left(\sum_{k=1}^{K^\alpha}\frac{C_k^{2\overline{\delta}}}{2\pi k}+\frac{C^{\overline{\delta}}}{\kappa-\lambda_\delta\beta_1}\left(1-e^{-(\kappa-\lambda_\delta\beta_1)t}\right)+\sum_{k=1}^{K^\alpha}\frac{1}{2\pi k}\left(C_k^{1\overline{\delta}}\sin(2\pi kt)-C_k^{2\overline{\delta}}\cos(2\pi kt)\right)\right)}, \tag{1.2.25}$$

$$\overline{\delta}(t) = \tfrac{\alpha_0+\lambda_\delta\beta_2}{\kappa-\lambda_\delta\beta_1} + \left(\overline{\delta}_0 - \tfrac{\alpha_0+\lambda_\delta\beta_2}{\kappa-\lambda_\delta\beta_1} - \sum_{k=1}^{K^\alpha} C_k^{1\overline{\delta}}\right)e^{-(\kappa-\lambda_\delta\beta_1)t} + \sum_{k=1}^{K^\alpha} C_k^{1\overline{\delta}}\cos(2\pi kt) + C_k^{2\overline{\delta}}\sin(2\pi kt). \tag{1.2.26}$$

The constants in Equations (1.2.25)-(1.2.26) are given by

$$C^{\bar{\delta}} = \bar{\delta}_0 - \tfrac{\alpha_0 + \lambda_\delta \beta_2}{\kappa - \lambda_\delta \beta_1} - \sum_{k=1}^{K^\alpha} C_k^{1\bar{\delta}},$$

$$C_k^{1\bar{\delta}} = \tfrac{2(\kappa - \lambda_\delta \beta_1)}{(\kappa - \lambda_\delta \beta_1)^2 + 4\pi^2 k^2} ((\kappa - \lambda_\delta \beta_1)\alpha_k^{(1)} - 2\pi k \alpha_k^{(2)}),$$

$$C_k^{2\bar{\delta}} = \tfrac{2(\kappa - \lambda_\delta \beta_1)}{(\kappa - \lambda_\delta \beta_1)^2 + 4\pi^2 k^2} (2\pi k \alpha_k^{(1)} + (\kappa - \lambda_\delta \beta_1)\alpha_k^{(2)}), \quad k = 1, \dots, K^\alpha.$$

Proof.

The solution to the ODEs (1.2.23) and (1.2.24) can be expressed as

$$\bar{S}(t) = \bar{S}_0 e^{\int_0^t \left(r + \lambda_S \beta_2 - (1 - \beta_1 \lambda_S)\bar{\delta}(s)\right) ds}, \tag{1.2.27}$$

$$\bar{\delta}(t) = \left(\bar{\delta}_0 + \int_0^t e^{-(\lambda_\delta \beta_1 - \kappa)s} (\lambda_\delta \beta_2 + \alpha_T(s)) ds\right) e^{(\lambda_\delta \beta_1 - \kappa)t}. \tag{1.2.28}$$

Calculating the integral term contained in Equation (1.2.28), we then obtain

$$\bar{\delta}(t) = \bar{\delta}_0 e^{-(\kappa - \lambda_\delta \beta_1)t} + \tfrac{\alpha_0 + \lambda_\delta \beta_2}{\kappa - \lambda_\delta \beta_1}\left(1 - e^{-(\kappa - \lambda_\delta \beta_1)t}\right) + \left(\sum_{k=1}^{K^\alpha} \alpha_k^{(1)} \left[f_k^{(1)}(s)\right]_{s=0}^{s=t} + \alpha_k^{(2)} \left[f_k^{(2)}(s)\right]_{s=0}^{s=t}\right) e^{-(\kappa - \lambda_\delta \beta_1)t}, \tag{1.2.29}$$

where

$$f_k^{(1)}(s) = \begin{cases} \tfrac{-(\kappa - \lambda_\delta \beta_1)}{\pi k} \sin(2\pi k s) e^{(\kappa - \lambda_\delta \beta_1)T} & ; \kappa - \lambda_\delta \beta_1 < 0 \\ \tfrac{2(\kappa - \lambda_\delta \beta_1)}{(\kappa - \lambda_\delta \beta_1)^2 + 4\pi^2 k^2} (2\pi k \sin(2\pi k s) + (\kappa - \lambda_\delta \beta_1)\cos(2\pi k s)) e^{(\kappa - \lambda_\delta \beta_1)s} & ; \kappa - \lambda_\delta \beta_1 > 0 \end{cases},$$

$$f_k^{(2)}(s) = \begin{cases} \tfrac{\kappa - \lambda_\delta \beta_1}{\pi k} \cos(2\pi k s) e^{(\kappa - \lambda_\delta \beta_1)T} & ; \kappa - \lambda_\delta \beta_1 < 0 \\ \tfrac{2(\kappa - \lambda_\delta \beta_1)}{(\kappa - \lambda_\delta \beta_1)^2 + 4\pi^2 k^2} ((\kappa - \lambda_\delta \beta_1)\sin(2\pi k s) - 2\pi k \cos(2\pi k s)) e^{(\kappa - \lambda_\delta \beta_1)s} & ; \kappa - \lambda_\delta \beta_1 > 0 \end{cases},$$

for $k = 1, \dots, K^\alpha$, $\kappa - \lambda_\delta \beta_1 \neq 0$ (see Appendix C for the calculation of the integral term). Since we are interested in the case that the instantaneous convenience yields must be bounded on the time interval $[0, \infty)$. This implies that $(\kappa - \lambda_\delta \beta_1)$ must be positive. The explicit formula for $\bar{\delta}(t)$ is obtained by substituting $f_k^{(i)}, i = 1, 2$, case $\kappa - \lambda_\delta \beta_1 > 0$, into Equation (1.2.29). Next, plugging the explicit formula of $\bar{\delta}(t)$ into Equation (1.2.27) and calculating the integral term give the explicit formula of $\bar{S}(t)$ as written in Equation (1.2.25).

□

The following proposition determines the futures prices under the assumption that the convenience yields are deterministic, i.e., setting $\delta_t \equiv \overline{\delta}_t(t)$, $\sigma_\delta = 0$ and excluding ρ from the model (1.2.1).

Proposition 3.

Suppose $\overline{\delta}(t) > \frac{-\beta_2}{\beta_1}$ for all $t \in [0,T]$. Then, under the no-arbitrage assumptions in a futures market and the convenience yields are deterministic, the futures price on day $t = 0$ with maturity date T, denoted by $\overline{F}_0^T$, satisfies

$$\overline{F}_0^T = E_{\mathbb{Q}}\left[S_T \mid \mathcal{F}_0\right] = \overline{S}(T)\,|_{\overline{S}_0 = S_0} \,. \qquad (1.2.30)$$

Proof.

By setting $\delta_t \equiv \overline{\delta}(t)$, $\sigma_\delta = 0$ in the model (1.2.1), the dynamics of S_t can be written as

$$dS_t = (r + \lambda_S\beta_2 - (1 - \lambda_S\beta_1)\overline{\delta}(t))S_t\,dt + \sqrt{\beta_1\overline{\delta}(t) + \beta_2}\,S_t\,dW_t^{(1)}, \qquad (1.2.31)$$

for $0 < t \leq T$. Since $\overline{\delta}(t) > \frac{-\beta_2}{\beta_1}$ for all $t \in [0,T]$, so the volatilities of the commodity spot prices have meaningful. The linear stochastic differential equation (1.2.31) has an explicit solution in the following form:

$$S_t = S_0\, e^{\int_0^t a(s)ds - \frac{1}{2}\int_0^t b^2(s)ds + \int_0^t b(s)dW_s^{(1)}}, \qquad (1.2.32)$$

where $a(t) := r + \lambda_S\beta_2 - (1 - \lambda_S\beta_1)\overline{\delta}(t)$ and $b(t) := \sqrt{\beta_1\overline{\delta}(t) + \beta_2}$.

Taking the conditional expectation with respect to the probability measure $\mathbb{Q}$ on the both sides of Equation (1.2.32) yields

$$E_{\mathbb{Q}}\left[S_T \mid \mathcal{F}_0\right] = S_0\, e^{\int_0^T a(s)ds}\, E_{\mathbb{Q}}[\, e^{-\frac{1}{2}\int_0^T b^2(s)ds + \int_0^T b(s)dW_s^{(1)}} \mid \mathcal{F}_0\,]. \qquad (1.2.33)$$

Using the result of Proposition 3.5.12 in Karatzas-Shreve (1988) [K-01], we have the conditional expectation on the RHS of Equation (1.2.33) is equal to one. Note that the remaining term is similar to the term on the RHS of Equation (1.2.27) in which t and $\overline{S}_0$ are replaced by T and S_0, respectively. Thus, the proof is now complete. □

From Proposition 3, one can show without difficulty that, under the no-arbitrage assumptions, the futures price on day $t \in [0,T]$ maturity at date T denoted by $\overline{F}_t^T$, satisfies

$$\overline{F}_t^T = E_{\mathbb{Q}}\left[S_T \mid \mathcal{F}_t\right] = \overline{S}(T-t)\,|_{\overline{S}_0 = S_t} \,. \qquad (1.2.34)$$

We now return to the explicit formula of $\overline{\delta}(t)$ as expressed in Equation (1.2.26). It should be mentioned about the behavior of the instantaneous convenience yields at a long-run maturity. It is easy to see that the seasonal effects indicated by the periodic terms contained in the equation, have a strong influence to the convenience yields as time approaches infinity. Without these periodic terms, as $t \to \infty$, the instantaneous convenience yield converges to

$$\overline{\delta}_{\infty} := \frac{\alpha_0 + \lambda_\delta \beta_2}{\kappa - \lambda_\delta \beta_1}. \tag{1.2.35}$$

The second term on the RHS of Equation (1.2.26) have a small influence to the convenience yields at as time approaches infinity and it can be neglected under this consideration. It is not difficult to show that the third term on the LHS of Equation (1.2.26) is bounded with some constant $c_1 > 0$. This implies, for a sufficiently large time t_δ,

$$\overline{\delta}_{\infty} - c_1 \leq \overline{\delta}(t) \leq \overline{\delta}_{\infty} + c_1 \quad \text{for all } t \geq t_\delta.$$

In other words, at a large time, the instantaneous convenience yields oscillate within the range. In terms of commodity spot prices, it can be noticed from Equation (1.2.25) that the commodity spot prices behave in the same way as the instantaneous convenience yields in the case that the interest rate satisfies

$$r = \overline{r}_{\infty} := (1 - \lambda_S \beta_1)\overline{\delta}_{\infty} - \lambda_S \beta_2, \tag{1.2.36}$$

In this case, for a sufficiently large time t_S, the commodity spot prices oscillate within the range:

$$\overline{S}_{\infty} - c_2 \leq \overline{S}(t) \leq \overline{S}_{\infty} + c_2 \quad \text{for all } t \geq t_S,$$

and for some $c_2 > 0$, where

$$\overline{S}_{\infty} := \overline{S}_0 \, e^{-(1-\lambda_S \beta_1)\left(\frac{C^{\overline{\delta}}}{\kappa - \lambda_\delta \beta_1} + \sum_{k=1}^{K^\alpha} \frac{C_k^{2\overline{\delta}}}{2\pi k}\right)}. \tag{1.2.37}$$

On the other hand, the commodity spot prices move up (down) away from $\overline{S}_{\infty}$ as time approaches infinity in the case that the interest rate is higher (lower) than $\overline{r}_{\infty}$. The graphs of $\overline{\delta}(t)$ and $\overline{S}(t)$ with these three different cases of parameters settings are shown in Figure 1.6 with $K^\alpha = 2$ and the initial conditions $\overline{\delta}_0 = 0, \overline{S}_0 = 50$.

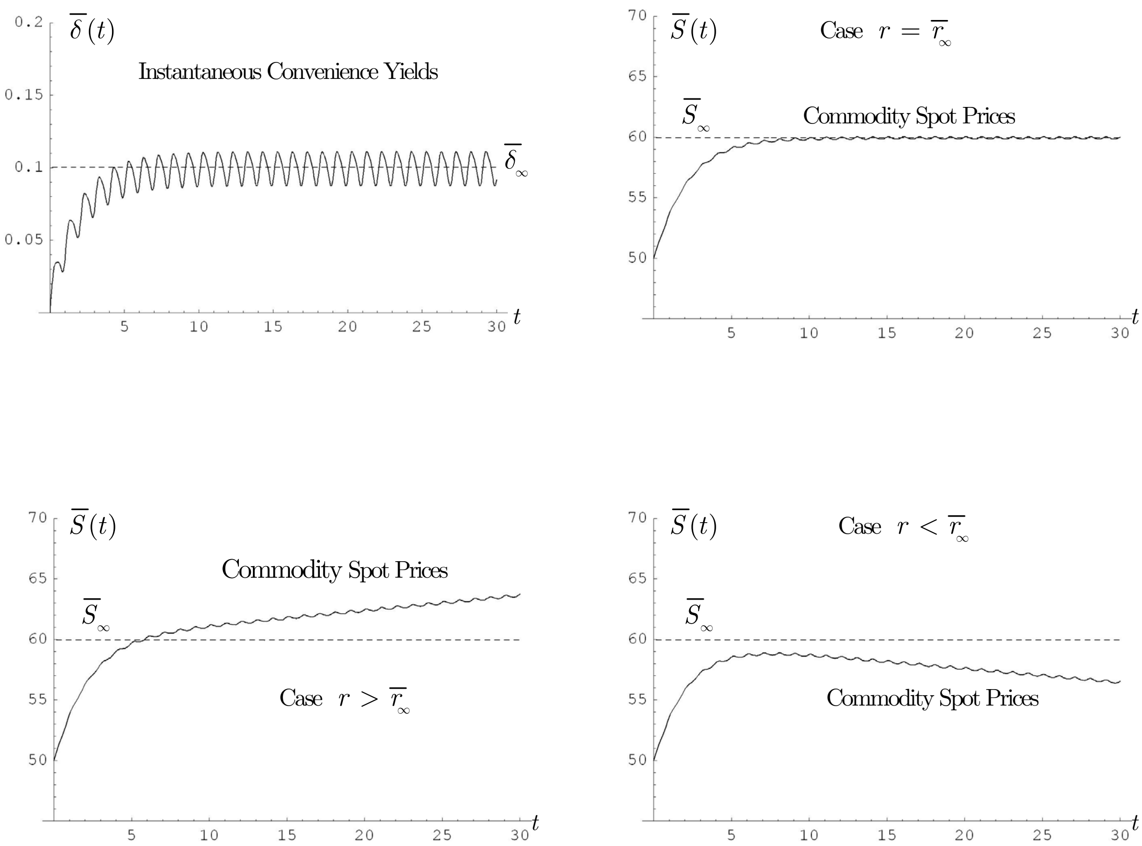


Figure 1.6: The convenience yields and the spot prices under the absence of randomness

1.3 Valuation of Commodity Derivatives

In this section, we derive a partial differential equation for commodity futures prices under the no-arbitrage assumptions. Furthermore, closed-form solutions for no-arbitrage futures prices are presented and their properties are investigated. From the closed-form solutions, we have the logarithm of a no-arbitrage futures price is a linear-affine function of the logarithm of a commodity spot price and an instantaneous convenience yield. Therefore, we can uniquely determine the commodity spot price and the instantaneous convenience yield from two no-arbitrage futures prices having different maturities. Applying the Itô formula to the closed-form solutions, we derive the dynamics of logarithmic futures prices process which will be used in Chapter 2 for estimation of the unknown parameters. In the last subsection, we derive partial differential equations for the prices of European futures options. The futures options pricing formulas are presented in which the option prices can be obtained by using traditional numerical techniques for solving the systems of ODEs and evaluating the improper integrals.

1.3.1 Partial Differential Equation for Futures Prices

Let $T > 0$ be a fixed time horizon and D a domain in $\mathbb{R}^m$, i.e., an open connected subset of $\mathbb{R}^m$. Under a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t\in[t_0,T]}$, we consider the SDE

$$dX_t = b(t, X_t)dt + \sum_{j=1}^{m} \sigma_j(t, X_t) dW_t^{(j)}, \quad X_{t_0} = x_0 \in D, \tag{S1}$$

for $t_0 \in [0,T)$, and $t \in (t_0, T]$, with continuous functions $b : [0,T] \times D \to \mathbb{R}^m$, and $\sigma_j : [0,T] \times D \to \mathbb{R}^m$, $j = 1, \ldots, m$, where $W \equiv (W_t^{(1)}, \ldots, W_t^{(m)})^{\mathrm{T}}$ denotes an m-dimensional Brownian motion. We write b and each σ_j as a $(1\times m)$ column vector and we define the $(m \times m)$ matrix-valued function σ by $\sigma_{ij} := (\sigma_j)_i$. The operator $\tilde{\mathcal{A}}$ on sufficient smooth functions $f : [0,T] \times D \to \mathbb{R}$ is defined by

$$(\tilde{\mathcal{A}} f)(t,x) := \frac{1}{2} \sum_{i,j=1}^{m} a_{ij}(t,x) \frac{\partial^2 f}{\partial x_i \partial x_j}(t,x) + \sum_{i=1}^{m} b_i(t,x) \frac{\partial f}{\partial x_i}(t,x), \tag{S2}$$

where $\mathbf{A}(t,x) \equiv (a_{ij}(t,x))$ denotes the matrix-valued function called the *diffusion matrix* for which each component is given by

$$a_{ij}(t,x) = \sum_{k_1,k_2=1}^{m} \sigma_{ik_1}(t,x) \rho_{k_1k_2} \sigma_{jk_2}(t,x), \text{ for } 1 \le i,j \le m,$$

with the correlation $\rho_{k_1k_2}$ satisfies $dW_t^{(k_1)} dW_t^{(k_2)} = \rho_{k_1k_2} dt$, for all $1 \le k_1, k_2 \le m$. For given a measurable function $h : D \to [0,\infty)$ and $\tilde{r} \ge 0$, we define $u : [0,T] \times D \to [0,\infty]$ by

$$u(t,x) := E_{\mathbb{P}}[e^{-\tilde{r}(T-t)} h(X_T) \mid \mathcal{F}_t] = E_{\mathbb{P}}[e^{-\tilde{r}(T-t)} h(X_T) \mid X_t = x]. \tag{S3}$$

Note that $u(t,x)$ is well define in $[0,\infty]$ if $X \equiv (X_s)_{s\in[t,T]}$ with $X_t = x$, does not explode or leave D before T, $\mathbb{P}$- a.s.. The following theorem is useful for deriving partial differential equations for futures and option prices.

Theorem 1.3.1. (Application of Theorem 1 in Heath-Schweizer (2000) [H-01])
Suppose that the sufficient conditions on X, D, b, and σ, imposed in Theorem 1 of Heath-Schweizer (2000) [H-01], hold, i.e.,

(A1) The coefficients b and σ_j, $j = 1, \ldots, m$, are on $[0,T] \times D$ locally Lipschitz-continuous in x, uniformly in t, i.e., for each compact subset F of D, there is a positive constant K_F such that

$$|G(t, y_1) - G(t, y_2)| \le K_F |y_1 - y_2|$$

for all $t \in [0,T]$, $y_1, y_2 \in F$, and for all $G \in \{b, \sigma_1, \ldots, \sigma_m\}$.

(A2) For all $(t_0, x_0) \in [0,T) \times D$, *the solution* $X \equiv (X_t)_{t\in[t_0,T]}$ *of the SDE (S1) with* $X_{t_0} = x_0$, *neither explodes nor leaves D before T,* $\mathbb{P}$*- a.s., i.e.,*

$$\mathbb{P}\left[\sup_{t_0\le t\le T}\left|X_t\right| < \infty\right] = 1 \text{ and } \mathbb{P}\left[X_t \in D, \forall t \in [t_0,T]\right] = 1.$$

(A3) There exists a sequence $(D_n)_{n\in\mathbb{N}}$ *of bounded domains contained in D such that* $\bigcup_{n=1}^{\infty} D_n = D$ *and such that for each n, the partial differential equation (PDE)*

$$\frac{\partial w}{\partial t} + \tilde{A}w - \tilde{r}w = 0 \text{ in } (0,T)\times D_n,$$

with boundary condition $w(t,x) = u(t,x)$ *on* $(0,T)\times\partial D_n \cup \{T\}\times D_n$ *has a classical solution* $w_n(t,x)$.

Then, u, defined in (S3), is in $C^{1,2}((0,T)\times D)$ *and satisfies the PDE*

$$\frac{\partial u}{\partial t} + \tilde{A}u - \tilde{r}u = 0 \quad \text{in } (0,T)\times D, \tag{1.3.1}$$

with boundary condition

$$u(T,x) = h(x) \quad \text{for } x \in D. \tag{1.3.2}$$

Moreover, there exists a unique classical solution to the PDE (1.3.1) and (1.3.2).

Remark 1.1.

The assumption (A3) is implied by the combination of (A3′), (A3a′), (A3b′), and (A3e′), which are imposed in Heath-Schweizer (2000) [H-01] as follows.

(A3′) There exists a sequence $(D_n)_{n\in\mathbb{N}}$ of bounded domains with $\bar{D}_n \subseteq D$ such that $\bigcup_{n=1}^{\infty} D_n = D$, each D_n has a C^2-boundary and for each n,

(A3a′) the (drift) function b and the (diffusion) matrix-valued function $\mathbf{A}(t,x)$ are uniformly Lipschitz-continuous on $[0,T]\times\bar{D}_n$,

(A3b′) $\mathbf{A}(t,x)$ is uniformly elliptic on $\mathbb{R}^m$ for all $(t,x)\in[0,T)\times D_n$, i.e., there is $\delta_n > 0$ such that $(y, \mathbf{A}(t,x)y^{\mathrm{T}})_{\mathbb{R}^m} \ge \delta_n |y|^2$ for all $y = (y_1, y_2, \ldots, y_m) \in \mathbb{R}^m$, and

(A3e′) u is finite and continuous on $[0,T]\times\partial D_n \cup \{T\}\times\bar{D}_n$.

Remark 1.2.

(1) Since $\mathbf{A}(t,x)$ is symmetric and has only real elements. Therefore, if the smallest real eigenvalue of $\mathbf{A}(t,x)$ is greater than zero for each $(t,x)\in[0,T)\times D$, we have (A3b′).

(2) Assume that (A1) and (A2) hold. If h is continuous and satisfies $|h(y)| \le M(1+|y|)$ for all $y \in D$ where M is a positive constant, then we have (A3e′).

By applying Theorem 1.3.1, we are now able to derive a partial differential equation for the futures prices under the no-arbitrage assumptions.

Proposition 4. (PDE for Futures Prices)
Under the no-arbitrage assumptions in a futures market, the futures price of a commodity at time t with maturity date T, denoted by $F^T \equiv F^T(t, S_t, \delta_t)$, *must equal to the expected value of its underlying commodity spot price at the maturity date T under the equivalent martingale measure* $\mathbb{Q}$, *i.e.,*

$$F^T(t, S_t, \delta_t) = E_{\mathbb{Q}}\left[S_T \mid \mathcal{F}_t\right]. \qquad (1.3.3)$$

Furthermore, by supposing that the condition (1.2.10) in Proposition 1 holds, then we can apply Theorem 1.3.1 to the dynamics of the commodity spot prices and the instantaneous convenience yields as expressed in the SDEs (1.2.1) with $X \equiv (S, \delta)$, $\tilde{r} = 0$, $u \equiv F^T$, *and* $h \equiv S$. *This implies* F^T *is in* $C^{1,2}(U_T)$ *and satisfies the PDE*

$$\frac{\partial F^T}{\partial t} + \frac{1}{2}(\beta_1\delta + \beta_2)S^2\frac{\partial^2 F^T}{\partial S^2} + \frac{1}{2}\sigma_\delta^2(\beta_1\delta + \beta_2)\frac{\partial^2 F^T}{\partial \delta^2} + \rho\sigma_\delta(\beta_1\delta + \beta_2)S\frac{\partial^2 F^T}{\partial\delta\partial S}$$
$$+(r - \delta + \lambda_S(\beta_1\delta + \beta_2))S\frac{\partial F^T}{\partial S} + (\alpha_T(t) - \kappa\delta + \lambda_\delta(\beta_1\delta + \beta_2))\frac{\partial F^T}{\partial \delta} = 0 \qquad (1.3.4)$$

in $U_T := (0, T) \times D$, *subject to the terminal condition*

$$F^T(T, S, \delta) = S \ \text{ for } (S, \delta) \text{ in } D, \qquad (1.3.5)$$

where $D := (0, \infty) \times (\frac{-\beta_2}{\beta_1}, \infty)$.

Proof.

It remains to show that, under the condition (1.2.10), the assumptions (A1), (A2), and (A3) hold. These results imply the existence and uniqueness of the classical solution which coincides F^T in U_T. Consider the process $X := (S_t, \delta_t)$. From the SDEs (1.2.1), we have the drift and the diffusion coefficients of S_t and δ_t are C^1 in (t, x) on U_T, then it is clear that (A1) is satisfied. The results obtained from Proposition 1 imply (A2). In order to verify (A3), we must show that (A3′), (A3a′), (A3b′), and (A3e′) hold. We set

$$U_T = (0, T) \times \bigcup_{n=2}^{\infty} D_n,$$

where for integer $n \geq 2$, we define the domains

$$D_n := (\tfrac{1}{n}, n) \times (\tfrac{-\beta_2}{\beta_1} + \tfrac{1}{n}, n)$$

with smoothed corners so that (A3′) is satisfied. One can easily see that, for all n, the drift and the diffusion coefficients of S_t and δ_t are C^1 in (t, x) on $U_T^{(n)} := [0,T] \times \overline{D}_n$, (A3a′) is clear. Since $h(S,\delta) = S$ satisfies the conditions in (2) of Remark 1.2, so we have (A3e′). Namely, $F^T(t,S,\delta)$ is finite and continuous in U_T. To achieve (A3b′), we verify (1) of Remark 1.2. For each $(t,S,\delta) \in U_T^{(n)}$, we compute the eigenvalues of the diffusion matrix and then we obtain the smallest real eigenevalue:

$$\lambda^*(t,S,\delta) = \tfrac{1}{4}(\beta_1\delta + \beta_2)\left((S^2 + \sigma_\delta^2) - \sqrt{S^4 + 2(2\rho^2 - 1)S^2\sigma_\delta^2 + \sigma_\delta^4}\right).$$

Since $\rho \in (-1,1)$, then $\lambda^* > 0$ for all $(t,S,\delta) \in U_T^{(n)}$ and for all n. This implies (A3b′) and the proof is now complete. □

1.3.2 Closed-Form Solutions for Futures Prices

We showed in Proposition 4 that the PDE (1.3.4) subject to the terminal condition (1.3.5) has a unique classical solution. In this subsection, we derive the solution by following the approach used in Nielsen-Schwartz (2004) [N-02]. We obtain closed-form solutions for no-arbitrage futures prices. In the case that the risk premiums and the seasonal parameters are zero: $\lambda_S = 0, \lambda_\delta = 0,$ and $\alpha_T(t) = \alpha_0$, the solution is similar to the one developed by Nielsen-Schwartz (2004) [N-02] which is basically the special case of our model. The proof of the following proposition is provided in Appendix D.

Proposition 5. (Determination of Futures Prices)
For given and fixed maturity date T, the solution to the PDE (1.3.4) subject to the terminal condition (1.3.5) can be expressed as

$$F^T(t, S_t, \delta_t; \theta) = S_t\, e^{A(T-t;\theta) + B(T-t;\theta)\,\delta_t}, \tag{1.3.6}$$

for all $(t, S_t, \delta_t) \in U_T$, where θ is the vector of all unknown parameters, and for all $\tau \geq 0$,

$$A(\tau;\theta) = (r + \lambda_S\beta_2)\tau + \left((\lambda_\delta + \rho\sigma_\delta)\beta_2 + \alpha_0\right)\Big[f_1(s;\theta)\Big]_{s=0}^{s=\tau} + \tfrac{1}{2}\sigma_\delta^2\beta_2\left[f_2(s;\theta)\right]_{s=0}^{s=\tau}$$
$$+\sum_{k=1}^{K^\alpha}\left(\alpha_k^{(1)}\left[f_c(s,T,k;\theta)\right]_{s=0}^{s=\tau} + \alpha_k^{(2)}\left[f_s(s,T,k;\theta)\right]_{s=0}^{s=\tau}\right), \tag{1.3.7}$$

$$B(\tau;\theta) = -\frac{2\,(1-p_3)\left(e^{\sqrt{p}\tau} - 1\right)}{\left(\sqrt{p} + p_2 + (\sqrt{p} - p_2)\,e^{\sqrt{p}\tau}\right)}. \tag{1.3.8}$$

The functions and the constants in Equations (1.3.7) – (1.3.8) are given by

$$f_1(s;\theta) = \frac{1}{2p_1}\left((\sqrt{p} - p_2)s - 2\ln\left(\sqrt{p} + p_2 + (\sqrt{p} - p_2)e^{\sqrt{p}s}\right)\right)$$

$$f_2(s;\theta) = \frac{1}{4p_1^2}\left((\sqrt{p} - p_2)^2 s + \frac{4\sqrt{p}(\sqrt{p} + p_2)}{\sqrt{p} + p_2 + (\sqrt{p} - p_2)e^{\sqrt{p}s}} + 4p_2 \ln\left(\sqrt{p} + p_2 + (\sqrt{p} - p_2)e^{\sqrt{p}s}\right)\right)$$

$$f_c(s,T,k;\theta) = -\frac{(1-p_3)\left(2\pi k\sqrt{p}\cos(2\pi k(T-s)) - \left((2\pi k)^2\left(e^{\sqrt{p}s} - 1\right) + pe^{\sqrt{p}s}\right)\sin(2\pi k(T-s))\right)}{\pi k\left((2\pi k)^2 + p\right)e^{\sqrt{p}s}}$$

$$f_s(s,T,k;\theta) = -\frac{(1-p_3)\left(\left((2\pi k)^2\left(e^{\sqrt{p}s} - 1\right) + pe^{\sqrt{p}s}\right)\cos(2\pi k(T-s)) + 2\pi k\sqrt{p}\sin(2\pi k(T-s))\right)}{\pi k((2\pi k)^2 + p)e^{\sqrt{p}s}}$$

$$p = p_2^2 + 4p_1 - 4p_1p_3,$$

$$p_1 = \tfrac{1}{2}\sigma_\delta^2\beta_1,\quad p_2 = (\lambda_\delta + \rho\sigma_\delta)\beta_1 - \kappa,\quad p_3 = \lambda_S\beta_1,$$

with the conditions $p > 0$ *and* $|p_2| < \sqrt{p}$ *which imply* $1 - p_3 > 0$.

It is easy to see that, from Equations (1.3.7)-(1.3.8), $A(0;\theta) = B(0;\theta) = 0$ and this implies the terminal condition (1.3.5). In Proposition 6, we show the properties and the estimates of the functions given in Proposition 5. The proof of Proposition 6 is given in Appendix E.

Proposition 6.

(1) $B(\tau;\theta)$ *is a nonpositive strictly decreasing function in* τ *on* $[0,\infty)$ *and satisfies the estimate*

$$|B(\tau;\theta)| \le \lim_{\tau'\to\infty}|B(\tau';\theta)| = \frac{2(1-p_3)}{\sqrt{p} - p_2},\quad \textit{for all } \tau \ge 0.$$

Moreover, for fixed $(t,S) \in [0,T)\times(0,\infty)$, *the mapping* $\delta \mapsto F^T(t,S,\delta;\theta)$ *is strictly decreasing and strictly convex on* $(\frac{-\beta_2}{\beta_1},\infty)$ *with* $\lim_{\delta\to\infty} F^T(t,S,\delta;\theta) = 0$. *This last assertion implies that the futures price tends to be lower than the spot price when the instantaneous convenience yield is sufficiently high.*

(2) $[f_1(s;\theta)]_{s=0}^{s=\tau}$ *is a nonpositive strictly decreasing function in* τ *on* $[0,\infty)$ *and* $[f_2(s;\theta)]_{s=0}^{s=\tau}$ *is a nonnegative strictly increasing function in* τ *on* $[0,\infty)$.

(3) $f_s(\tau,T,k;\theta)$ *and* $f_s(\tau;T;k;\theta)$ *satisfy the estimates*

$$|f_c(\tau,T,k;\theta)|, |f_s(\tau,T,k;\theta)| \le (1-p_3)\left(\frac{2\sqrt{p}}{(2\pi k)^2 + p} + \frac{1}{\pi k}\right) \le (1-p_3)\left(\frac{2\sqrt{p}}{(2\pi)^2 + p} + \frac{1}{\pi}\right),$$

for all $k = 1,2,\ldots,K^\alpha$, *and for all* $\tau \ge 0$.

In Figure 1.7, we show the graphs of $\left[f_1(s;\theta)\right]_{s=0}^{s=T-t}$, $\left[f_2(s;\theta)\right]_{s=0}^{s=T-t}$, $f_c(T-t,T,1;\theta)$, $f_s(T-t,T,1;\theta)$, $A(T-t;\theta)$, and $B(T-t;\theta)$, respectively, where $T=5$ and $t\in[0,5]$. Figure 1.8 illustrates the evolution of the futures prices obtained from the closed-form solution given in Proposition 5 at time $t=0$, $t=0.2, t=2$, $t=3, t=4,$ and $t=5$. The parameters follow Case 1 in Table 1.1 and (S_t,δ_t) varies within $(0,20]\times[-0.1,0.1]$. From Figure 1.8, one can observe seasonal variation in futures prices by considering the futures prices at the corner point $(S_t,\delta_t)=(20,-0.1)$ at time $t=0, t=0.2,$ and $t=2$.

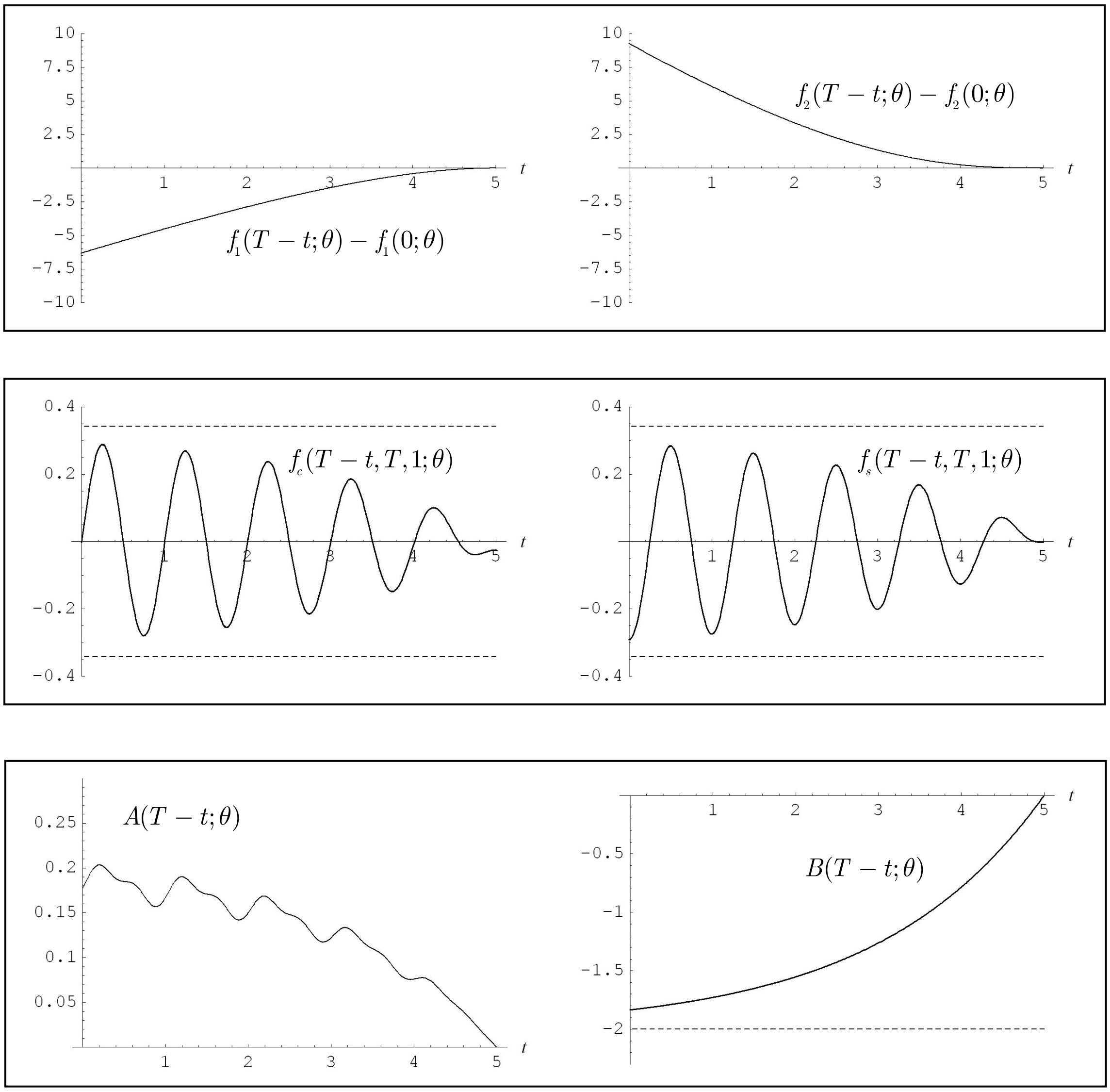


Figure 1.7: Graphs of all components of the closed-form solution

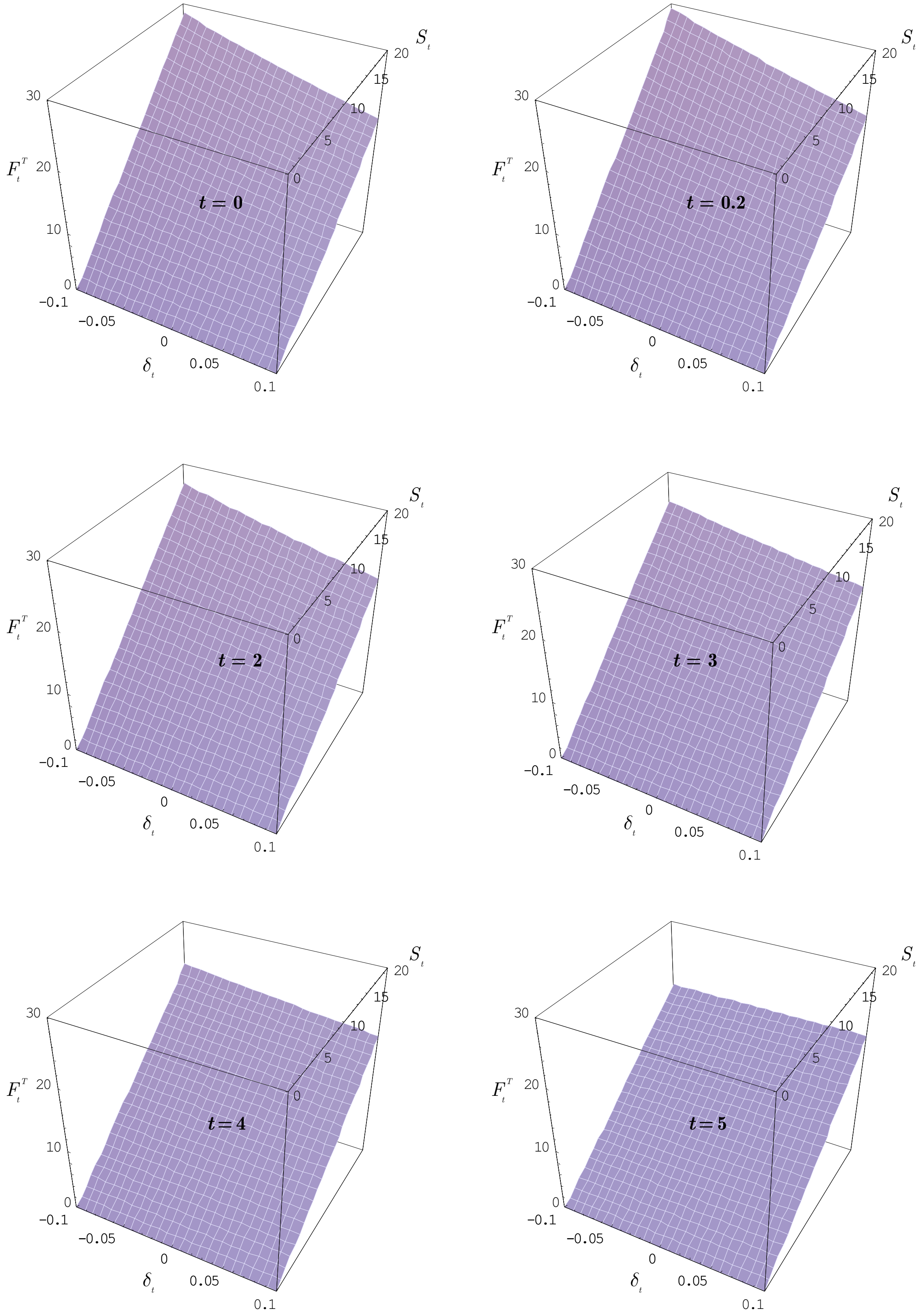


Figure 1.8: Evolution of the futures prices

It should be pointed out to the result obtained from Proposition 6 that futures prices tend to be lower than spot prices when convenience yields are sufficiently high. When the convenience yields reach a sufficient large negative value, on the other hand, one can easily verify that the futures prices tend to be higher than the spot prices. These results are consistent with the theory of storage as previously described. However, the convenience yields must be bounded from below as shown in Expression (1.2.6). Thus, we conclude from the formula (1.3.6) that the futures prices tend to be higher than the spot prices when the convenience yields approach $-\beta_2 / \beta_1$ where the maximum cost rate of carrying yields, β_2 / β_1, is allowed to be sufficiently high.

We next consider the effects of the instantaneous convenience yields on the futures prices. Suppose that T is fixed and $A(T-t;\theta)$ is ignored in this consideration. From (1) of Proposition 6, $B(T-t;\theta)$ has nonpositive values from day $t=0$ (the day that the futures contract is initiated) until the maturity date $t=T$ and the function is strictly increasing in variable t on $[0,T]$. This result indicates that the *direct effect* of the instantaneous convenience yields defined by

$$\psi_\delta^D(T-t,\delta_t;\theta) := B(T-t;\theta)\,\delta_t, \tag{1.3.9}$$

has the opposite sign of δ_t. This implies, on day t, if $\delta_t > 0$ then the futures price is lower than the spot price and vice versa.

We consider further the *indirect effect* of the instantaneous convenience yields on the futures prices defined by

$$\psi_\delta^I(T-t;\theta) := \sum_{i=1}^{5} \psi_\delta^{I_i}(T-t;\theta), \tag{1.3.10}$$

where

$$\psi_\delta^{I_1}(T-t;\theta) := \lambda_\delta \beta_2 \left[f_1(s;\theta)\right]_{s=0}^{s=T-t}, \ \psi_\delta^{I_2}(T-t;\theta) := \rho\sigma_\delta \beta_2 \left[f_1(s;\theta)\right]_{s=0}^{s=T-t},$$

$$\psi_\delta^{I_3}(T-t;\theta) := \alpha_0 \left[f_1(s;\theta)\right]_{s=0}^{s=T-t}, \quad \psi_\delta^{I_4}(T-t;\theta) := \tfrac{1}{2}\sigma_\delta^2\beta_2 \left[f_2(s;\theta)\right]_{s=0}^{s=T-t},$$

$$\psi_\delta^{I_5}(T-t;\theta) := \sum_{k=1}^{K^\alpha}\Big(\alpha_k^{(1)}\left[f_c(s,T,k;\theta)\right]_{s=0}^{s=T-t} + \alpha_k^{(2)}\left[f_s(s,T,k;\theta)\right]_{s=0}^{s=T-t}\Big),$$

which can be separated into five parts as above and it does not depend on the risk free interest rate r. It should be noted here that $\psi_\delta^I(\cdot;\theta)$ contains all parameters describing the dynamics of δ_t (including the seasonal parameters). From (2) of Proposition 6, we see that the signs of $\psi_\delta^{I_i}, i=1,2,3,$ have the opposite signs of $\lambda_\delta, \rho,$ and α_0, respectively, and $\psi_\delta^{I_4}$ is always nonnegative. These effects have strong influences to the futures prices as time to maturity increases. In addition, from (3) of Proposition 6, the seasonal effect term $\psi_\delta^{I_5}$ is bounded as time to maturity increases. The net effect of the instantaneous convenience yields to the futures prices is defined as follows:

$$\psi_\delta^{net}(T-t,\delta_t;\theta) := \psi_\delta^D(T-t,\delta_t;\theta) + \psi_\delta^I(T-t;\theta). \tag{1.3.11}$$

Next, we define the effect of risk free interest rate on the futures prices as follows:

$$\psi_r(T-t;r) := r\,(T-t) \geq 0. \tag{1.3.12}$$

The effect of market price of risk on the futures prices is defined by

$$\psi_{\lambda_S}(T-t;\theta) := \lambda_S \beta_2\,(T-t). \tag{1.3.13}$$

From Equations (1.3.9)-(1.3.13), the futures price can be written as

$$F^T(t,S_t,\delta_t;\theta) = S_t\, e^{\psi_r(T-t;r)+\psi_{\lambda_S}(T-t;\lambda_S,\beta_2)+\psi_\delta^I(T-t;\theta)+\psi_\delta^D(T-t,\delta_t;\theta)}. \tag{1.3.14}$$

It should be remarked here that, even though at any time t in which $\delta_t = 0$, or equivalently $\psi_\delta^D \equiv 0$, the futures price $F^T(t,S_t,0;\theta)$ still has influences from the indirect effects of the convenience yields, namely, the effects from $\psi_\delta^{I_i}, i = 1,\ldots,5$.

1.3.3 Extraction of Commodity Prices and Convenience Yields

As described in Introduction, commodity spot prices and instantaneous convenience yields cannot be observed in this setting. Only the corresponding futures prices are available in the futures market. In order to extract the spot prices and the instantaneous convenience yields from no-arbitrage futures prices, we need the following assumption.

Assumption A

In a futures market of a commodity, for every trading day t, we can observe two no-arbitrage futures prices $F_t^{T_1}$ and $F_t^{T_2}$ without measurement error where T_1 and T_2 are, respectively, the closest and the second-closest maturity dates of the corresponding futures contracts that have been traded on day t.

Suppose that Assumption A holds. Replacing $F_t^{T_i}, i = 1,2,$ into Equation (1.3.6) and taking logarithm to both sides of the equations give us a system of linear equations:

$$\left.\begin{aligned} \ln F_t^{T_1} &= \ln S_t + A(T_1-t;\theta) + B(T_1-t;\theta)\delta_t \\ \ln F_t^{T_2} &= \ln S_t + A(T_2-t;\theta) + B(T_2-t;\theta)\delta_t \end{aligned}\right\} \tag{1.3.15}$$

with respect to the two unknown variables $\ln S_t$ and δ_t. Solving the system (1.3.15) heuristically, we obtain

$$\ln S_t = \frac{(B(T_1 - t;\theta)\ln F_t^{T_2} - B(T_2 - t;\theta)\ln F_t^{T_1}) + G(t,T_1,T_2;\theta)}{B(T_1 - t;\theta) - B(T_2 - t;\theta)}, \qquad (1.3.16)$$

$$\delta_t = \frac{\left(\ln F_t^{T_1} - \ln F_t^{T_2}\right) + \left(A(T_2 - t;\theta) - A(T_1 - t;\theta)\right)}{B(T_1 - t;\theta) - B(T_2 - t;\theta)}, \qquad (1.3.17)$$

where

$$G(t,T_1,T_2;\theta) := A(T_1 - t;\theta)B(T_2 - t;\theta) - A(T_2 - t;\theta)B(T_1 - t;\theta). \qquad (1.3.18)$$

We know from (1) in Proposition 6 that $B(\tau;\theta)$ is a strictly decreasing function on $[0,\infty)$. This implies the denominators on the RHS of Equation (1.3.16) and Equation (1.3.17) never reach zero unless $T_1 = T_2$. Therefore, the formulas in Equation (1.3.16) and Equation (1.3.17) are well-defined.

The above derivation shows that we are able to extract or filter out the spot price S_t and the instantaneous convenience yield δ_t from two no-arbitrage futures prices having different maturity dates. We finish this subsection by writing S_t and δ_t as functions of $(t, F_t^{T_1}, F_t^{T_2};\theta)$ in the following forms:

$$S(t,F_t^{T_1},F_t^{T_2};\theta) \equiv \exp\left(\frac{(B(T_1 - t;\theta)\ln F_t^{T_2} - B(T_2 - t;\theta)\ln F_t^{T_1}) + G(t,T_1,T_2;\theta)}{B(T_1 - t;\theta) - B(T_2 - t;\theta)}\right), \qquad (1.3.19)$$

$$\delta(t,F_t^{T_1},F_t^{T_2};\theta) \equiv \frac{\left(\ln F_t^{T_1} - \ln F_t^{T_2}\right) + \left(A(T_2 - t;\theta) - A(T_1 - t;\theta)\right)}{B(T_1 - t;\theta) - B(T_2 - t;\theta)}, \qquad (1.3.20)$$

and then we obtain

$$F^T \equiv F^T(t,F_t^{T_1},F_t^{T_2};\theta) \equiv S(t,F_t^{T_1},F_t^{T_2};\theta)\, e^{A(T-t;\theta)+B(T-t;\theta)\,\delta(t,F_t^{T_1},F_t^{T_2};\theta)}. \qquad (1.3.21)$$

1.3.4 Dynamics of Log-Futures Prices

We first recall the Itô formula which will be used to derive the dynamics of futures prices.

Theorem (The Itô formula)

Let $(X_t, W_t) = ((X_t^{(1)},\dots,X_t^{(m)}),(W_t^{(1)},\dots,W_t^{(m)})^{\mathrm{T}})$, $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ *be an m-dimensional Itô diffusion process as described in Subsection 1.3.1,* $\tilde{\mathcal{A}}$ *be the corresponding operator defined in (S2) within the subsection, and* $f:[0,T]\times D \to \mathbb{R}$ *be a function which belongs to* $C^{1,2}([0,T]\times D)$. *Then* $F(t,\omega) = f(t, X_t(\omega))$ *is an Itô diffusion process and its dynamics is given by*

$$dF_t = \left(\frac{\partial f}{\partial t}(t,X_t) + (\tilde{\mathcal{A}}f)(t,X_t)\right)dt + \sum_{i=1}^{m}\left(\sum_{j=1}^{m}\sigma_{ij}(t,X_t)\ dW_t^{(j)}\right)\frac{\partial f}{\partial x_i}(t,X_t).$$

Applying the Itô formula to the futures prices as expressed in Equation (1.3.6) and using Equation (1.3.4) which eliminates the drift term of the futures prices process, we then obtain the dynamics of the (no-arbitrage) futures prices under the probability measure $\mathbb{Q}$:

$$dF_t^T = \sqrt{\beta_1\delta_t + \beta_2}\, F_t^T dW_t^{(1)} + \sigma_\delta \sqrt{\beta_1\delta_t + \beta_2}\, B(T-t;\theta)\, F_t^T\, dW_t^{(2)}, \tag{1.3.22}$$

for $t \in [0,T]$. Figure 1.10 shows a sample path of futures prices and a sample path of spot prices obtained by simulations via Milstein scheme with Case 1 of parameters setting in Table 1.1 and the maturity date $T = 1$. The simulations are taken over the time interval $[0,T]$ with the initial values $S_0 = 10$, $\delta_0 = 0.01$, $F_0^T = F^T(0, S_0, \delta_0; \theta)$, and the number of time step $M = 300$. It should be noticed from the figure that, under this case, the futures prices are higher than the spot prices on $[0,T)$. This implies the basis is positive before the maturity date T. Namely, the cost-of-carry yield plus the implicit interest charge exceeds the gross convenience yield during that time interval.

Consider the dynamics of futures prices in Equation (1.3.22). Since δ_t cannot be observed in this setting. This makes parameter estimation procedures are very complicated. Fortunately, we can extract δ_t from two futures prices with different maturities by using the extraction formula (1.3.20). In other words, we can assume that δ_t is a deterministic function of time. Let X_t^T be the logarithm of the futures price, i.e., $X_t^T \equiv \ln F_t^T$. Applying the Itô formula to the process (1.3.22), we have the process $(X_t^T)_{t\in[0,T]}$ satisfies the SDE:

$$dX_t^T = \sqrt{\beta_1\delta(t, F_t^{T_1^t}, F_t^{T_2^t}; \theta) + \beta_2}\ dW_t^{(1)} + \sigma_\delta \sqrt{\beta_1\delta(t, F_t^{T_1^t}, F_t^{T_2^t}; \theta) + \beta_2}\ B(T-t;\theta)\, dW_t^{(2)}. \tag{1.3.23}$$

In Chapter 2, we use the process $(X_t^T)_{t\in[0,T]}$ as the underlying process to construct the approximate MLEs for estimation of the unknown parameters.

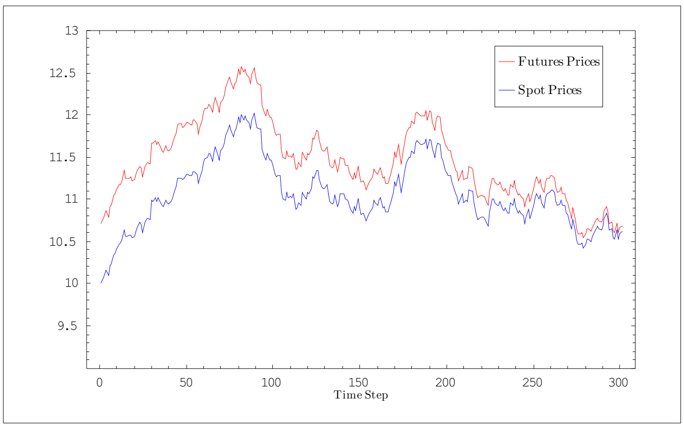


Figure 1.9: Futures Prices and Spot Prices Case 1

1.3.5 Valuation of European Futures Options

In this subsection, we consider European options written on a commodity futures contract. We first consider a European call option written on a commodity futures contract and the following notations are employed: t is a current time, $T > 0$ is the maturity date of the commodity futures contract, $T_c \in [t, T)$ is the maturity date of the call option written on the commodity futures contract, K is the strike price of the option, $C(t, F_t^T, \delta_t; T_c, T, K; \theta)$ is the value at time t of the call option that expires at time T_c written on the futures contract that expires at time T.

Corollary 1.3.1. (PDE for Call Futures Option Prices)

Under the no-arbitrage assumptions in a futures market, the call option price must equal to the present value of the expected payoff of the call option under the equivalent martingale measure $\mathbb{Q}$, i.e.,

$$C(t, F_t^T, \delta_t; T_c, T, K; \theta) = E_{\mathbb{Q}}\left[e^{-r(T_c - t)} \max\left(0, F_{T_c}^T - K\right) \mid \mathcal{F}_t\right]. \tag{1.3.24}$$

Furthermore, by supposing that the condition (1.2.10) in Proposition 1 and the following additional condition holds:

$$0 < \zeta(t; \theta) < 1 \ \ \textit{for all } t \in [0, T), \tag{A4}$$

where $\zeta(t;\theta) := 1 + 2\rho\sigma_\delta B(T-t;\theta) + \sigma_\delta^2 B^2(T-t;\theta)$, then we can apply Theorem 1.3.1 to the dynamics of the futures prices as expressed in the SDE (1.3.22) and the dynamics of the instantaneous convenience yields as expressed in the SDE (1.2.1) with $X \equiv (F^T, \delta)$, $\tilde{r} = r$, $u \equiv C$, and $h \equiv \max(0, F^T - K)$. This implies C is in $C^{1,2}\left(U_{T_c}\right)$ and satisfies the PDE

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma_F^2(T-t,\delta;\theta)F^2\frac{\partial^2 C}{\partial F^2} + \frac{1}{2}\sigma_\delta^2(\beta_1\delta + \beta_2)\frac{\partial^2 C}{\partial \delta^2}$$
$$+\sigma_{F\delta}(T-t,\delta;\theta)F\frac{\partial^2 C}{\partial F \partial \delta} + \mu_\delta(t,\delta;\theta)\frac{\partial C}{\partial \delta} - rC = 0 \tag{1.3.25}$$

in $U_{T_c} := (0, T_c) \times D$, subject to the terminal condition

$$C(T_c, F^T, \delta; T_c, T, K; \theta) = \max\left(0, F^T - K\right) \ \textit{for } (F^T, \delta) \textit{ in } D, \tag{1.3.26}$$

where F in Equation (1.3.25) denotes F^T and $D := (0,\infty) \times (\frac{-\beta_2}{\beta_1}, \infty)$. The functions in Equation (1.3.25) are given by

$$\sigma_F^2(\tau,\delta;\theta) = (\beta_1\delta + \beta_2)\big(1 + 2\rho\sigma_\delta B(\tau;\theta) + \sigma_\delta^2 B^2(\tau;\theta)\big), \tag{1.3.27}$$

$$\sigma_{F\delta}(\tau,\delta;\theta) = (\beta_1\delta + \beta_2)\big(\rho\sigma_\delta + \sigma_\delta^2 B(\tau;\theta)\big), \tag{1.3.28}$$

$$\mu_\delta(\tau,\delta;\theta) = \alpha_T(\tau) - \kappa\delta + \lambda_\delta\,(\beta_1\delta + \beta_2)\ \textit{for all}\ \tau \geq 0. \tag{1.3.29}$$

Proof.

This proof is analogous to the proof of Proposition 4. Consider the process $X := (F_t^T, \delta_t)$. From the SDE (1.3.22), we have the drift and the diffusion coefficients of F_t are C^1 in (t,x) on U_{T_c}. Then (A1) is satisfied. From Proposition (5), we have

$$F_t^T = S_t \exp(A(T-t) + B(T-t)\delta_t).$$

Since (S_t, δ_t) neither explodes nor leaves D before T a.s. and, by the above formula of F_t^T, so is $X = (F_t^T, \delta_t)$. Thus, we have (A2). Next, we set

$$U_{T_c} = (0, T_c) \times \bigcup_{n=2}^{\infty} D_n,$$

where for integer $n \geq 2$, we define the domains

$$D_n := (\tfrac{1}{n}, n) \times (\tfrac{-\beta_2}{\beta_1} + \tfrac{1}{n}, n)$$

with smoothed corners so that (A3′) is satisfied. Because the drift and diffusion coefficients of F_t^T and δ_t are C^1 in $U_{T_c}^{(n)} := [0, T_c] \times \bar{D}_n$ for all n, thus (A3a′) is obvious. Since $h(F^T, \delta) = \max(0, F^T - K)$ satisfies the conditions in (2) of Remark 1.2, hence, we have (A3e′). Namely, $C(t, F^T, \delta)$ is finite and continuous in U_{T_c}. Next, we verify (A3b′) by using (1) of Remark 1.2. The smallest real eigenvalue of the diffusion matrix of the process X is of the following form:

$$\lambda^*(t, F^T, \delta) = \tfrac{1}{4}(\beta_1\delta + \beta_2)\Big(((F^T)^2\zeta(t;\theta) + \sigma_\delta^2) - \sqrt{(F^T)^4\zeta^2(t;\theta) + 2(\sigma_\delta F^T)^2\xi(t;\theta) + \sigma_\delta^4}\Big),$$

where $\xi(t;\theta) := \zeta(t;\theta) + 2(\rho^2 - 1)$. From condition (A4), we have $\xi(t;\theta) < (2\rho^2 - 1)\zeta(t;\theta)$ which implies that λ^* is always positive for all $(t, F^T, \delta) \in U_{T_c}^{(n)}$ for all n, and this complete the proof. □

The PDE (1.3.25) is more general than the one derived by Heston (1993) [H-02]. In Heston model, $\sigma_F^2(\cdot)$, $\sigma_{F\delta}(\cdot)$, and $\mu_\delta(\cdot)$ are constants and the solutions are available in closed-forms. By following his approach, we obtain the solutions which can be implemented by using some traditional numerical schemes such as the Runge-Kutta methods for solving the systems of ODEs and the Gauss-Laguerre Quadrature formula for evaluating the approximate values of the improper integrals. The following proposition is proved in Appendix F.

Proposition 7. (Determination of Call Futures Option Prices)

For given and fixed maturity dates T, T_c, and a strike price K, the solution to the PDE (1.3.25) subject to the terminal condition (1.3.26) can be expressed as

$$C(t, F^T, \delta; T_c, T, K; \theta) = e^{-r(T_c - t)}\left(F^T P_1 - K P_2\right), \tag{1.3.30}$$

for $(t, F^T, \delta) \in U_{T_c}$, *where* $P_j \equiv P_j(t, x, \delta; T_c, T, \ln K; \theta)$, $j = 1,2$, *having values in* $[0,1]$, *are condition probabilities which can be expressed as*

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-\mathbb{i}\phi \ln K}\varphi_j(\phi; t, x, \delta; T_c, T; \theta)}{\mathbb{i}\phi}\right] d\phi, \tag{1.3.31}$$

where $\mathbb{i} = \sqrt{-1}$, $x \equiv \ln F^T$, *and* φ_j, $j = 1,2$, *are characteristic functions.*

The characteristic functions are given by

$$\varphi_j(\phi; t, x, \delta; T_c, T; \theta) = e^{\tilde{A}_j(\phi; T_c - t, T_c, T; \theta) + \tilde{B}_j(\phi; T_c - t, T_c, T; \theta)\,\delta + \mathbb{i}\phi x}, \tag{1.3.32}$$

where $\tilde{A}_j, \tilde{B}_j$, $j = 1,2$, *solve the systems of ordinary differential equations:*

$$\tilde{A}_j'(\phi;\tau,T_c,T;\theta) - \tfrac{1}{2}\beta_2\,\sigma_\delta^2\,\tilde{B}_j^2(\phi;\tau,T_c,T;\theta) - \tilde{C}_{\tilde{A}_j}(\phi;\tau,T_c,T;\theta)\tilde{B}_j(\phi;\tau,T_c,T;\theta) - \tilde{D}_{\tilde{A}_j}(\phi;\tau,T_c,T;\theta) = 0$$

$$\tilde{B}_j'(\phi;\tau,T_c,T;\theta) - \tfrac{1}{2}\beta_1\,\sigma_\delta^2\tilde{B}_j^2(\phi;\tau,T_c,T;\theta) - \tilde{C}_{\tilde{B}_j}(\phi;\tau,T_c,T;\theta)\,\tilde{B}_j(\phi;\tau,T_c,T;\theta) - \tilde{D}_{\tilde{B}_j}(\phi;\tau,T_c,T;\theta) = 0$$

for $\tau > 0$, $\;{}' = \dfrac{d}{d\tau}$, (1.3.33)

subject to the initial conditions $\tilde{A}_j(\phi; 0, T_c, T; \theta) = 0$ *and* $\tilde{B}_j(\phi; 0, T_c, T; \theta) = 0$.

The functions in the ODEs (1.3.33) are given by

$$\tilde{C}_{\tilde{A}_j}(\phi;\tau,T_c,T;\theta) = (\mathbb{i}\phi + 2 - j)\beta_2\left(\rho\,\sigma_\delta + \sigma_\delta^2 B(\tau^*;\theta)\right) + \alpha_T(T_c - \tau) + \lambda_\delta\beta_2,$$

$$\tilde{C}_{\tilde{B}_j}(\phi;\tau,T_c,T;\theta) = (\mathbb{i}\phi + 2 - j)\beta_1\left(\rho\,\sigma_\delta + \sigma_\delta^2 B(\tau^*;\theta)\right) + \lambda_\delta\beta_1 - \kappa,$$

$$\tilde{D}_{\tilde{A}_j}(\phi;\tau,T_c,T;\theta) = -\tfrac{1}{2}\phi\left(\phi + (-1)^j\mathbb{i}\right)\beta_2\left(1 + 2\rho\sigma_\delta B(\tau^*;\theta) + \sigma_\delta^2 B^2(\tau^*;\theta)\right),$$

$$\tilde{D}_{\tilde{B}_j}(\phi;\tau,T_c,T;\theta) = -\tfrac{1}{2}\phi\left(\phi + (-1)^j\mathbb{i}\right)\beta_1\left(1 + 2\rho\sigma_\delta B(\tau^*;\theta) + \sigma_\delta^2 B^2(\tau^*;\theta)\right),$$

for $j = 1,2$, $\tau \geq 0$, *where* $\tau^* = T - T_c + \tau$.

We next consider a European put option written on a commodity futures contract. We denote further that $T_p \in [t, T)$ is the maturity date of a put option written on the commodity futures contract, $P(t, F^T, \delta; T_p, T, K; \theta)$ is the value at time t of the put option that expires at time T_p written on the futures contract that expires at time T.

Corollary 1.3.2. (PDE for Put Futures Option Prices)
Under the no-arbitrage assumptions in a futures market, the put option price must equal to the present value of the expected payoff of the put option under the equivalent martingale measure $\mathbb{Q}$, *i.e.,*

$$P(t, F_t^T, \delta_t; T_p, T, K; \theta) = E_{\mathbb{Q}}\left[e^{-r(T_p - t)} \max\left(0, K - F_{T_p}^T\right) \mid \mathcal{F}_t\right]. \tag{1.3.34}$$

Furthermore, by supposing that the conditions imposed in Corollary 1.3.1 hold, then we can apply Theorem 1.3.1 to the dynamics of the futures prices as expressed in the SDE (1.3.22) and the dynamics of the instantaneous convenience yields as expressed in the SDE (1.2.1) with $X \equiv (F^T, \delta)$, $\tilde{r} = r$, $u \equiv P$, *and* $h \equiv \max(0, K - F^T)$. *This implies P is in* $C^{1,2}\left(U_{T_p}\right)$ *and satisfies the PDE (1.3.25), with replacing C by P, in* $U_{T_p} := (0, T_p) \times D$, *subject to the terminal condition*

$$P(T_p, F^T, \delta; T_p, T, K; \theta) = \max\left(0, K - F^T\right) \text{ for } (F^T, \delta) \text{ in } D. \tag{1.3.35}$$

Proof.
The proof of this corollary is similar to the proof of Corollary 1.3.1 in which we consider $h(F^T, \delta) = \max(0, K - F^T)$ which satisfies the conditions in (2) of Remark 1.2. □

The value of the put futures options can be evaluated by using the following proposition which is proved in Appendix F.

Proposition 8. (Determination of Put Futures Option Prices and the Put-Call Parity)
For given and fixed maturity dates T, T_p, *and a strike price K, the solution to the PDE (1.3.25), with replacing C by P, subject to the terminal condition(1.3.35) can be expressed as*

$$P(t, F^T, \delta; T_p, T, K; \theta) = e^{-r(T_p - t)}\left(K(1 - P_2) - F^T(1 - P_1)\right), \tag{1.3.36}$$

for $(t, F^T, \delta) \in U_{T_p}$, *where* $P_j \equiv P_j(t, x, \delta; T_p, T, \ln K; \theta), j = 1,2,$ *having values in* $[0,1]$, *are the condition probabilities as expressed in Proposition 7. Moreover, if* $T_p = T_c = \tilde{T}$, *then we obtain the put-call parity:*

$$P - C = e^{-r(\tilde{T} - t)}(K - F^T), \tag{1.3.37}$$

for $(t, F^T, \delta) \in U_{\tilde{T}} := (0, \tilde{T}) \times D$.

We now return to Proposition 7. To evaluate the call futures option prices, we have to solve the systems of ODEs written in Equation (1.3.33) which contains the functions of complex variables. Therefore, we express $\tilde{A}_j$ and $\tilde{B}_j$ for $j = 1, 2,$ of the following forms:

$$\tilde{A}_j(\phi;\tau,T_c,T;\theta) = \tilde{A}_{1j}(\phi;\tau,T_c,T;\theta) + \tilde{A}_{2j}(\phi;\tau,T_c,T;\theta)\mathbb{i}, \tag{1.3.38}$$

$$\tilde{B}_j(\phi;\tau,T_c,T;\theta) = \tilde{B}_{1j}(\phi;\tau,T_c,T;\theta) + \tilde{B}_{2j}(\phi;\tau,T_c,T;\theta)\mathbb{i}, \tag{1.3.39}$$

where A_{kj} and $B_{kj}, k = 1, 2,$ are real-valued functions to be determined. For each j, applying Equation (1.3.38) and Equation (1.3.39) to Equation (1.3.33) give us a new system of ODEs of A_{kj} and B_{kj} with the initial conditions $A_{kj}(\phi;0,T_c,T;\theta) = 0$ and $B_{kj}(\phi;0,T_c,T;\theta) = 0,$ $k = 1, 2.$ In Appendix G, the systems of ODEs are derived explicitly.

Pricing futures options is important because these types of options are now traded on many different exchanges and popular among those financial derivatives. In the case of commodity, the most popular contracts include those on corn, soybean, sugar, wheat, crude oil, heating oil, natural gas, and gold. The reasons why people choose to trade options on futures rather than options on the underlying commodity can be described as follows. The main reason appears to be that a futures contract is more liquid and easier to trade than the underlying commodity. The futures prices are known immediately from trading on the futures exchange, whereas the spot prices of the commodity may not be so readily available. Furthermore, it is much easier and more convenient to make or take delivery of commodity futures contract than it is to make or take delivery of the underlying commodity. A final point is that commodity futures options tend to entail lower transactions costs than commodity spot options in many situations.

Finally, we close this chapter by illustrating the evolutions of call futures option prices written on a commodity futures contract obtained by the numerical implementations in Appendix F. By using the put-call parity expressed in Proposition 8, the corresponding put futures option prices are obtained. Figure 1.11 shows the evolution of the call futures option prices and the evolution of the put futures option prices at $t = 0, t = 0.5,$ and $t = \tilde{T} = 1,$ where $T_c = T_p = \tilde{T}.$ The parameters follow Case 1 in Table 1.1 with the strike price $K = 11,$ and the futures contract expires at $T = 2.$

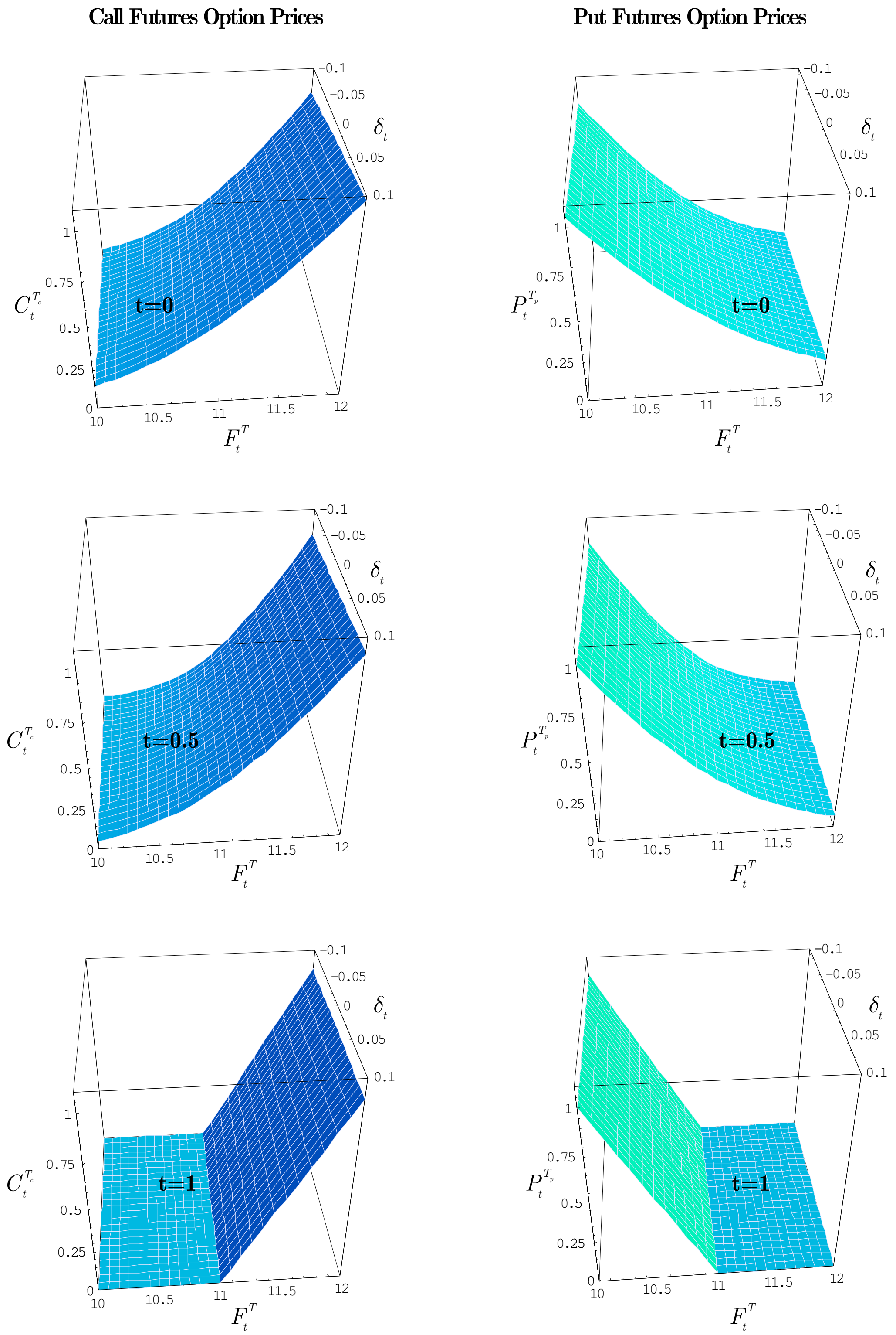


Figure 1.10: Evolution of European Futures Option Prices of Commodity

Chapter 2

Transition Density of Log-Futures Prices and Approximate Maximum Likelihood Estimators

In this chapter, we consider the transition density of the log-futures prices process written in Equation (1.3.23). We solve the forward Kolmogorov equation to obtain the forward transition density of the process. Due to the obtained forward transition density contains integral terms of a discretely observed function, we derive a closed-form approximation of the forward transition density using observed futures prices data and the error estimate is investigated. Using the closed-form approximation, we construct a sequence of the approximate log-likelihood functions of the logarithmic futures prices data and prove its convergence in probability to the true log-likelihood function. This convergence implies that the limit of the sequence of the approximate MLEs is close to the exact MLEs which can be inferred to the true-parameters describing the dynamics of the process. The results obtained in this chapter will be used in Chapter 3 for the calibration of model (M) introduced in Introduction.

2.1 Transition Density of Log-Futures Prices

The transition density of a diffusion process plays a central role in estimation of diffusion parameters from a discretely observed sample data of the diffusion process based on the method of maximum likelihood. Let X_t be an Itô diffusion process and $\tilde{p}_X(t,y;\mathbf{s},\mathbf{x};\theta)$ be the forward transition density of X_t given $X_{\mathbf{s}} = \mathbf{x}$ where θ is the vector of parameters. Suppose that we have discretely sampled observations of X_t at different time points: $t_1, t_2, \ldots, t_N$. The log-likelihood function of the observations $X_{t_1}, X_{t_2}, \ldots, X_{t_N}$, is just the sum of $\ln \tilde{p}_X\left(t_i, X_{t_i}; t_{i-1}, X_{t_{i-1}}; \theta\right)$, $i = 2,3,\ldots,N$. The vector of maximum likelihood estimators (MLEs) of unknown parameters is referred to a vector that maximizes the log-likelihood function over a parameter space Θ.

In this present section, we consider the forward transition density $\tilde{p}_{X^T}(t,y;\mathbf{s},\mathbf{x};\theta)$ of the logarithmic futures prices process:

$$dX_t^T = \sqrt{\beta_1\delta(t, F_t^{T_1^t}, F_t^{T_2^t};\theta) + \beta_2}\, dW_t^{(1)} + \sigma_\delta\sqrt{\beta_1\delta(t, F_t^{T_1^t}, F_t^{T_2^t};\theta) + \beta_2}\ B(T-t;\theta)\, dW_t^{(2)}, \tag{2.1.1}$$

under the equivalent martingale measure $\mathbb{Q}$, where θ is the vector of the model parameters, and $\mathbf{s} \in [0, T)$, $T > 0$, $\mathbf{x} \in \mathbb{R}$ are fixed. The forward transition density $\tilde{p}_{X^T}(t, y; \mathbf{s}, \mathbf{x}; \theta)$ satisfies the *forward Kolmogorov equation* :

$$\frac{\partial \tilde{p}_{X^T}}{\partial t}(t, y; \mathbf{s}, \mathbf{x}; \theta) - \frac{1}{2} a_T(t; \theta) \frac{\partial^2 \tilde{p}_{X^T}}{\partial y^2}(t, y; \mathbf{s}, \mathbf{x}; \theta) = 0 \quad \text{for}(t, y) \in (\mathbf{s}, T] \times \mathbb{R} \qquad (2.1.2)$$

subject to the condition

$$\lim_{t \downarrow \mathbf{s}} \tilde{p}_{X^T}(t, y; \mathbf{s}, \mathbf{x}; \theta) = \hat{\delta}(y - \mathbf{x}), \qquad (2.1.3)$$

where $\hat{\delta}(\cdot)$ is Dirac-delta function on $\mathbb{R}$ and the *diffusion coefficient of* X_t^T is given by

$$a_T(t; \theta) = (\beta_1 \delta(t, F_t^{T_1}, F_t^{T_2}; \theta) + \beta_2)\left(1 + 2\rho\sigma_\delta B(T - t; \theta) + \sigma_\delta^2 B^2(T - t; \theta)\right), \qquad (2.1.4)$$

and

$$\delta(t, F_t^{T_1}, F_t^{T_2}; \theta) = \frac{\left(\ln F_t^{T_1} - \ln F_t^{T_2}\right) + \left(A(T_2 - t; \theta) - A(T_1 - t; \theta)\right)}{B(T_1 - t; \theta) - B(T_2 - t; \theta)}.$$

Proposition 9.

For fixed θ, suppose that $a_T(t; \theta)$ is a positive continuous function in t on $[0, T]$. Then

$$\tilde{p}_{X^T}(t, y; \mathbf{s}, \mathbf{x}; \theta) = \frac{1}{\sqrt{2\pi \int_{\mathbf{s}}^{t} a_T(\eta; \theta) d\eta}} \exp\left(-\frac{(y - \mathbf{x})^2}{2 \int_{\mathbf{s}}^{t} a_T(\eta; \theta) d\eta} \right), \qquad (2.1.5)$$

is the solution of the forward Kolmogorov equation (1.2.2) subject to the condition (2.1.3).

Proof.

We first transform the variable t as follows:

$$\hat{t}(t) := \int_{\mathbf{s}}^{t} a_T(\eta; \theta) d\eta. \qquad (\text{T1})$$

It should be noted that $\hat{t}(t)$ is in C^1 and strictly increasing in t on $(\mathbf{s}, T]$. With this transformation, applying the chain rule to the partial derivatives in Equation (2.1.2) gives us the following PDE:

$$a_{_T}(t;\theta)\left(\frac{\partial \tilde{p}_{_{X^T}}}{\partial \hat{t}}(\hat{t},y:0,\mathbf{x};\theta)-\frac{1}{2}\frac{\partial^2 \tilde{p}_{_{X^T}}}{\partial y^2}(\hat{t},y:0,\mathbf{x};\theta)\right)=0,\ \text{for } (\hat{t},y)\in(0,\hat{T}]\times\mathbb{R}, \quad \text{(T2)}$$

where $\hat{T}=\hat{t}(T)$. Since $a_{_T}(t;\theta)>0$. Thus, the second factor on the LHS of (T2) must equal to zero, i.e.,

$$\frac{\partial \tilde{p}_{_{X^T}}}{\partial \hat{t}}(\hat{t},y:0,\mathbf{x};\theta)-\frac{1}{2}\frac{\partial^2 \tilde{p}_{_{X^T}}}{\partial y^2}(\hat{t},y:0,\mathbf{x};\theta)=0, \quad \text{for } (\hat{t},y)\in(0,\hat{T}]\times\mathbb{R}. \quad \text{(T3)}$$

The solution of (T3) is, in fact, the Gaussian transition density with mean $\mathbf{x}$ and variance $\hat{t}$, i.e.,

$$\tilde{p}_{_{X^T}}(\hat{t},y;0,\mathbf{x};\theta)=\frac{1}{\sqrt{2\pi\hat{t}}}\exp\left(-\frac{(y-\mathbf{x})^2}{2\hat{t}}\right), \text{ for } (\hat{t},y)\in(0,\hat{T}]\times\mathbb{R}. \quad (2.1.6)$$

and we have

$$\lim_{\hat{t}\downarrow 0}\tilde{p}_{_{X^T}}(\hat{t},y;0,\mathbf{x};\theta)=\hat{\delta}(y-\mathbf{x}). \quad (2.1.7)$$

Equations (2.1.6)-(2.1.7) imply, respectively, $\tilde{p}_{_{X^T}}(t,y;\mathbf{s},\mathbf{x};\theta)$ as expressed in Equation (2.1.5) satisfies the forward Kolmogorov equation (2.1.2) and the condition (2.1.3). □

Suppose that Assumption A holds. Hence, the diffusion coefficient $a_{_T}(\cdot;\theta)$ can be observed only at the endpoints of the time interval $[\mathbf{s},t]$, namely, only $a_{_T}(\mathbf{s};\theta)$ and $a_{_T}(t;\theta)$ can be observed in this setting. To approximate the integral terms in Equation (2.1.5), we employ the Trapezoidal formula, i.e.,

$$\int_{\mathbf{s}}^{t} a_{_T}(\eta;\theta)\,d\eta \approx \frac{1}{2}(t-\mathbf{s})(a_{_T}(t;\theta)+a_{_T}(\mathbf{s};\theta)).$$

This leads to an approximate forward transition density of the logarithmic futures prices process

$$\tilde{p}^{A}_{_{X^T}}(t,y;\mathbf{s},\mathbf{x};\theta):=\frac{1}{\sqrt{\pi(t-\mathbf{s})\tilde{a}_{_T}(t;\mathbf{s};\theta)}}\exp\left(-\frac{(y-\mathbf{x})^2}{(t-\mathbf{s})\tilde{a}_{_T}(t;\mathbf{s};\theta)}\right), \quad (2.1.8)$$

where

$$\tilde{a}_{_T}(t;\mathbf{s};\theta):=a_{_T}(t;\theta)+a_{_T}(\mathbf{s};\theta).$$

It should be noted that the diffusion coefficient $a_{_T}(t;\theta)$ as given in Equation (2.1.4) should be continuous on $[0,T]$ but nowhere differentiable. Suppose for a moment that $a_{_T}(t;\theta)$ depended on the Itô diffusion process δ_t as written in Equation (2.1.4) with replacing $\delta(t,\cdot)$ by δ_t. Applying the Itô formula to $a_{_T}(t;\theta)$, its dynamics could be written as

$$da_{T}(t;\theta)=\mu_{a}^{T}(t,\delta_{t};\theta)dt+\sigma_{a}^{T}(t,\delta_{t};\theta)dW_{t}^{(2)},\quad t\in[0,T],\tag{2.1.9}$$

for some functions $\mu_{a}^{T}(t,\delta_{t};\theta)$ and $\sigma_{a}^{T}(t,\delta_{t};\theta)$. Suppose that $\mathbf{x}$ and y lie on a path of the logarithmic futures prices process (2.1.1), i.e.,

$$X_{\mathbf{s}}^{T}(\omega)=\mathbf{x}\ \text{ and }\ X_{t}^{T}(\omega)=y,\ \ \omega\in\Omega,$$

for $0\leq\mathbf{s}<t\leq T$, where Ω is the sample space. For a fixed parameters vector θ, we have the following error estimate.

Proposition 10.

$$\left|\ln\tilde{p}_{X^{T}}(t,y;\mathbf{s},\mathbf{x};\theta)-\ln\tilde{p}_{X^{T}}^{A}(t,y;\mathbf{s},\mathbf{x};\theta)\right|\leq C(t-\mathbf{s})^{\frac{1}{2}},\tag{2.1.10}$$

where C is a positive constant.

Proof.

We omit writing the argument θ in this proof. First, we consider

$$\ln\tilde{p}_{X^{T}}(t,y;\mathbf{s},\mathbf{x})-\ln\tilde{p}_{X^{T}}^{A}(t,y;\mathbf{s},\mathbf{x})$$

$$=\left(-\frac{1}{2}\ln\left(2\pi\int_{\mathbf{s}}^{t}a_{T}(\eta)d\eta\right)-\frac{(y-\mathbf{x})^{2}}{2\int_{\mathbf{s}}^{t}a_{T}(\eta)d\eta}\right)+\left(\frac{1}{2}\ln\left(\pi(t-\mathbf{s})\tilde{a}_{T}(t;\mathbf{s})\right)+\frac{(y-\mathbf{x})^{2}}{(t-\mathbf{s})\tilde{a}_{T}(t;\mathbf{s})}\right)$$

$$=\frac{1}{2}\ln\left(\frac{(t-\mathbf{s})(a_{T}(t)+a_{T}(\mathbf{s}))}{2\int_{\mathbf{s}}^{t}a_{T}(\eta)d\eta}\right)+\left(\frac{1}{(t-\mathbf{s})(a_{T}(t)+a_{T}(\mathbf{s}))}-\frac{1}{2\int_{\mathbf{s}}^{t}a_{T}(\eta)d\eta}\right)(y-\mathbf{x})^{2}$$

$$=\frac{1}{2}\ln\left(\frac{a_{T}(t)+a_{T}(\mathbf{s})}{2a_{T}(\tilde{\eta}_{1})}\right)+\left(\frac{2a_{T}(\tilde{\eta}_{1})-(a_{T}(t)+a_{T}(\mathbf{s}))}{2(t-\mathbf{s})a_{T}(\tilde{\eta}_{1})(a_{T}(t)+a_{T}(\mathbf{s}))}\right)(y-\mathbf{x})^{2},\ \text{ for some }\tilde{\eta}_{1}\in[\mathbf{s},t],$$

(applying mean value theorem to the integrals)

$$=\frac{1}{2}\ln\left(\frac{a_{T}(\tilde{\eta}_{2})}{a_{T}(\tilde{\eta}_{1})}\right)+\frac{(a_{T}(\tilde{\eta}_{1})-a_{T}(\tilde{\eta}_{2}))}{2(t-\mathbf{s})a_{T}(\tilde{\eta}_{1})a_{T}(\tilde{\eta}_{2})}(y-\mathbf{x})^{2},\ \text{ for some }\tilde{\eta}_{2}\in[\mathbf{s},t],$$

(using continuity of $a_{T}(\cdot)$ on $[\mathbf{s},t]$).

The above verification implies that

$$\left|\ln \tilde{p}_{X^T}(t,y;\mathbf{s},\mathbf{x}) - \ln \tilde{p}^A_{X^T}(t,y;\mathbf{s},\mathbf{x})\right| \le \frac{1}{2}\left|\ln a_T(\tilde{\eta}_2) - \ln a_T(\tilde{\eta}_1)\right| + \frac{\left|a_T(\tilde{\eta}_1) - a_T(\tilde{\eta}_2)\right|}{2(t-\mathbf{s})a_T(\tilde{\eta}_1)a_T(\tilde{\eta}_2)}(y-\mathbf{x})^2, \tag{2.1.11}$$

for some $\tilde{\eta}_1, \tilde{\eta}_2 \in [\mathbf{s}, t]$. Since $\mathbf{x}$ and y lie on a path of the Itô diffusion process $(X^T_t)_{t\in[0,T]}$ which is a Hölder continuous path with exponent γ close to ½ and we have

$$(y-\mathbf{x})^2 = \left(X^T_t(\omega) - X^T_{\mathbf{s}}(\omega)\right)^2 \le C_1(t-\mathbf{s})^{2\gamma}, \tag{2.1.12}$$

as $t \downarrow \mathbf{s}$ and for some constant $C_1 > 0$. Similarly, from the Itô diffusion process (2.1.9), we assume that the diffusion coefficient $a_T(\cdot)$ and $\ln a_T(\cdot)$ satisfy the following estimates:

$$\left|a_T(t_1) - a_T(t_2)\right| \le C_2(t_2 - t_1)^{\frac{1}{2}} \le C_2(t-\mathbf{s})^{\frac{1}{2}}, \tag{2.1.13}$$

$$\left|\ln a_T(t_1) - \ln a_T(t_2)\right| \le C_3(t_2 - t_1)^{\frac{1}{2}} \le C_3(t-\mathbf{s})^{\frac{1}{2}}, \tag{2.1.14}$$

for some constants $C_2, C_3 > 0$ and for any $\mathbf{s} \le t_1 \le t_2 \le t$. Applying (2.1.12)-(2.1.14) to (2.1.11), we obtain the estimate as expressed in (2.1.10) by choosing

$$C = \frac{C_3}{2} + \frac{C_1 C_2}{2\left(\inf\limits_{\tau\in[\mathbf{s},T]} a_T(\tau)\right)^2}. \qquad \square$$

The estimate (2.1.10) implies that when t approaches $\mathbf{s}$ the error estimate tends to zero, in other words, $\tilde{p}^A_{X^T} \to \tilde{p}_{X^T}$ as $t \downarrow \mathbf{s}$. This result will be used in the proof of the convergence in probability of the approximate log-likelihood function of log-futures prices data to the true log-likelihood function (Proposition 11).

Suppose that Assumption A holds and the true-parameters are known. Thus, we can calculate the diffusion coefficient $a_T(t_n;\theta)$ on day t_n by using Equation (2.1.4). For example, setting the parameters follow Case 1 in Table 1.1, we use the daily futures prices data of WR5 futures $\left\{(F^{T_1^n}_{t_n}, F^{T_2^n}_{t_n}) / n = 1,\ldots,365\right\}$ from AFET, the graph of piecewise line-approximation of the diffusion coefficient $a_T(t;\theta)$ of the logarithmic WR5 futures prices process is illustrated in Figure 2.1 (the upper graph) where $T = 1$, $\Delta = 1/365$, $t_n = (n-1)/365$, and $t_1 = 0$ which is started at August 26, 2004. The approximate forward transition density $\tilde{p}^A_{X^T}$ can be written as

$$\tilde{p}^{A}_{X^{T}}(t_n, X^T_t; t_{n-1}, \ln 10; \theta) = \frac{\exp\left(-\dfrac{365(X^T_t - \ln 10)^2}{\tilde{a}_T(t_n, F^{T_1^n}_{t_n}, F^{T_2^n}_{t_n}; t_{n-1}, F^{T_1^{n-1}}_{t_{n-1}}, F^{T_2^{n-1}}_{t_{n-1}}; \theta)}\right)}{\sqrt{\dfrac{\pi}{365}\tilde{a}_T(t_n, F^{T_1^n}_{t_n}, F^{T_2^n}_{t_n}; t_{n-1}, F^{T_1^{n-1}}_{t_{n-1}}, F^{T_2^{n-1}}_{t_{n-1}}; \theta)}}, n = 2, \ldots, 365,$$

in which we set the initial logarithmic futures price $X^T_{t_{n-1}} = \ln 10 \approx 2.30259$. Figure 2.1 also shows two examples of the graphs of $\tilde{p}^{A}_{X^{T}}$ on days $t_n, n = 141, 230$. We see from Figure 2.1 that the diffusion coefficient of the log-futures prices on day t_{230} is the highest one and on day t_{141} is the lowest one during the life time of the futures contract. This indicates that the probability of the log-futures price on day t_{230} move away far from the log-futures price of day t_{229} is greater than the probability of the log-futures price on day t_{141} move away far from the log-futures price of day t_{140} where we set $X^T_{t_{140}} = X^T_{t_{229}} = \ln 10$.

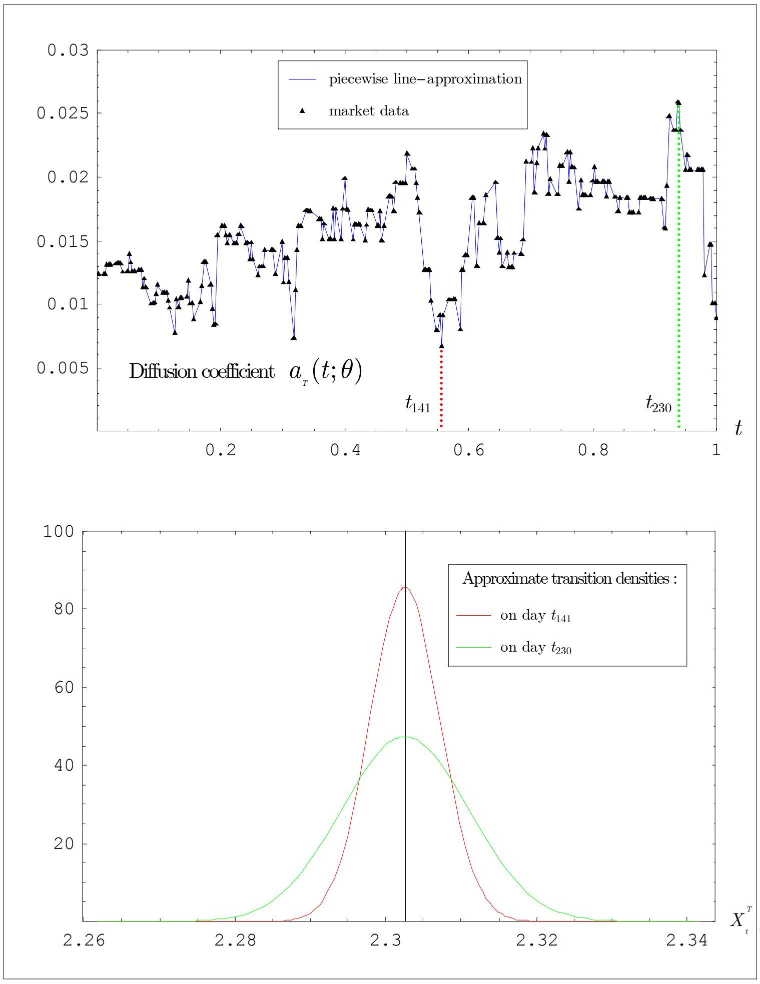


Figure 2.1: The diffusion coefficient and the approximate transition densities of logarithmic WR5 futures prices process on days t_{141} and t_{230} under Case 1

2.2 Approximate Maximum Likelihood Estimators

For a fixed maturity date T, we suppose that the logarithmic futures prices $X_{t_1}^T,\dots,X_{t_N}^T$ at the equidistant time points[4] $t_n, n=1,\dots,N$, on $[0,T]$, are observed without measurement error. Baye's rule combined with the Markovian nature of the process $(X_t^T)_{t\in[0,T]}$, which the discrete data inherit, implies that the log-likelihood function of the logarithmic futures prices data $X_N^T := \left\{X_{t_1}^T, X_{t_2}^T,\dots,X_{t_N}^T\right\}$, denoted by $\tilde{l}_N(\theta; X_N^T)$, has the simple form:

$$\tilde{l}_N(\theta; X_N^T) := \ln\left(\prod_{n=2}^{N} \tilde{p}_{X^T}(t_n, X_{t_n}^T; t_{n-1}, X_{t_{n-1}}^T; \theta)\right) = \sum_{n=2}^{N} \tilde{l}_{X^T}(t_n, X_{t_n}^T; t_{n-1}, X_{t_{n-1}}^T; \theta), \tag{2.2.1}$$

where $\tilde{l}_{X^T} \equiv \ln \tilde{p}_{X^T}$. From Equation (2.1.8), an approximate log-likelihood function of the logarithmic futures prices data X_N^T, denoted by $\tilde{l}_N^A(\theta; X_N^T, \Delta)$, can be constructed as follows:

$$\tilde{l}_N^A(\theta; X_N^T, \Delta) := \ln\left(\prod_{n=2}^{N} \tilde{p}_{X^T}^A(t_n, X_{t_n}^T; t_{n-1}, X_{t_{n-1}}^T, \Delta; \theta)\right) = \sum_{n=2}^{N} \tilde{l}_{X^T}^A(t_n, X_{t_n}^T; t_{n-1}, X_{t_{n-1}}^T, \Delta; \theta), \tag{2.2.2}$$

where $\tilde{l}_{X^T}^A \equiv \ln \tilde{p}_{X^T}^A$ and $\Delta > 0$ denotes the equidistant time step size.

Let the parameter space Θ be a compact subset of $\mathbb{R}^{n_\theta}$ for some positive integer n_θ and $\theta_0 \in \Theta^\circ$ denote the true-parameter describing the dynamics of X_t^T. For a fixed number of observations N and for any data X_N^T, we assume that the following assumptions hold.
For any $(t,y),(\mathbf{s},\mathbf{x}) \in [0,T]\times\mathbb{R}$,

(G1) the mapping $\theta \mapsto \tilde{l}_{X^T}(t,y;\mathbf{s},\mathbf{x};\theta)$ belongs to $C^3(\Theta)$,

(G2) the mapping $\theta \mapsto \tilde{l}_{X^T}^A(t,y;\mathbf{s},\mathbf{x};\Delta,\theta)$ belongs to $C^3(\Theta)$ for all $\Delta > 0$,

(G3) the mapping $\theta \mapsto \tilde{l}_N(\theta; X_N^T)$ has a unique maximizer $\tilde{\theta}_{N,X_N^T}^{MLE} \in \Theta^\circ$, i.e.,

$$\tilde{\theta}_{N,X_N^T}^{MLE} := \arg\max_{\theta\in\Theta} \tilde{l}_N(\theta; X_N^T), \tag{2.2.3}$$

(G4) the mapping $\theta \mapsto \tilde{l}_N^A(\theta; X_N^T, \Delta)$ has a unique maximizer $\tilde{\theta}_{N,X_N^T,\Delta}^A \in \Theta^\circ$ for all $\Delta > 0$, i.e.,

$$\tilde{\theta}_{N,X_N^T,\Delta}^A := \arg\max_{\theta\in\Theta} \tilde{l}_N^A(\theta; X_N^T, \Delta). \tag{2.2.4}$$

In fact, $\tilde{\theta}_{N,X_N^T}^{MLE}$ is the exact (but uncomputable) MLE for θ_0 and the vector $\tilde{\theta}_{N,X_N^T,\Delta}^A$ is called the "approximate MLE". Under the assumptions (G1)-(G4), we have the following proposition.

[4] The equidistant time points satisfy $0 \le t_1 < t_2 < \cdots < t_N \le T$ and $t_i - t_{i-1} = \Delta$ for all $i = 2,\dots,N$.

Proposition 11.

For a fixed number of observations N,

$$\sup_{\theta\in\Theta}\left|\tilde{l}_N^A(\theta;X_N^T,\Delta)-\tilde{l}_N(\theta;X_N^T)\right|\to 0 \ \textit{in}\ \mathbb{Q}_{\theta_0}\textit{ - probability as } \Delta\to 0^+ \tag{2.2.5}$$

and the approximate MLE sequence $\tilde{\theta}^A_{N,X_N^T,\Delta}$ *satisfies*

$$\left|\tilde{\theta}^A_{N,X_N^T,\Delta}-\tilde{\theta}^{MLE}_{N,X_N^T}\right|\to 0 \ \textit{in}\ \mathbb{Q}_{\theta_0}\textit{ - probability as } \Delta\to 0^+. \tag{2.2.6}$$

Proof.

Let $\mathbf{s}\geq 0$ and $\mathbf{x}\in\mathbb{R}$ for which $X_{\mathbf{s}}^T(\omega)=\mathbf{x}$ be fixed. Define

$$\tilde{r}_{X^T}(t,y;\mathbf{s},\mathbf{x},\Delta):=\sup_{\theta\in\Theta}\left|\tilde{l}_{X^T}^A(t,y;\mathbf{s},\mathbf{x};\Delta,\theta)-\tilde{l}_{X^T}(t,y;\mathbf{s},\mathbf{x};\theta)\right|, \tag{2.2.7}$$

for any $t\in(\mathbf{s},T]$ and $y\in\mathbb{R}$ for which $X_t^T(\omega)=y$, where $\Delta=t-\mathbf{s}$. Since Θ is compact and (G1)-(G2) hold. Thus, $\tilde{r}_{X^T}(t,y;\mathbf{s},\mathbf{x},\Delta)$ is well-defined for each $\Delta>0$. The convergence of (2.2.5) is obtained if we have the following convergence:

$$\tilde{r}_{X^T}\to 0 \ \text{in}\ \mathbb{Q}_{\theta_0}\text{ - probability as }\Delta\to 0^+. \tag{2.2.8}$$

Using the estimate (2.1.10) in Proposition 10, we have the estimate:

$$\begin{aligned}0\leq E_{\mathbb{Q}_{\theta_0}}\left[\tilde{r}_{X^T}(t,y;\mathbf{s},\mathbf{x},\Delta)\right]&=\int_{-\infty}^{\infty}\tilde{r}_{X^T}(t,y;\mathbf{s},\mathbf{x},\Delta)\,\tilde{p}_{X^T}(t,y;\mathbf{s},\mathbf{x};\theta_0)\,dy\\&\leq\int_{-\infty}^{\infty}\left(C\Delta^{\frac{1}{2}}\right)\tilde{p}_{X^T}(t,y;\mathbf{s},\mathbf{x};\theta_0)dy.\\&\leq C\Delta^{\frac{1}{2}},\end{aligned} \tag{2.2.9}$$

where $C>0$. The estimate (2.2.9) implies that the expectation of the error estimate $\tilde{r}_{X^T}$ converges to zero as $\Delta\to 0^+$ (equivalently, $t\downarrow\mathbf{s}$) under the probability measure $\mathbb{Q}_{\theta_0}$. Hence, by applying Chebyshev's inequality to the estimate (2.2.9), we obtain the convergence (2.2.8). By (G1)-(G4) and the proximity of the two objective functions, $\tilde{l}_N(\theta;X_N^T)$ and $\tilde{l}_N^A(\theta;X_N^T,\Delta)$, just obtained by the convergence (2.2.5), it follows from the standard argument that the approximate MLE sequence $\tilde{\theta}^A_{N,X_N^T,\Delta}$ converges to $\tilde{\theta}^{MLE}_{N,X_N^T}$ as $\Delta\to 0^+$ in the sense as the convergence (2.2.6).

□

Remark 2.1.
Expression (2.2.6) implies that, for a fixed sample size N, the approximate MLE $\tilde{\theta}^{A}_{N,X^T_N,\Delta}$ converges in $\mathbb{Q}_{\theta_0}$ - probability to the exact MLE $\tilde{\theta}^{MLE}_{N,X^T_N}$ as $\Delta \to 0^+$. Since the process $(X^T_t)_{t\in[0,T]}$ is non-stationary, the asymptotic properties of the MLE under stationary case cannot be applied. However, the basic features and concepts of the Fisher-Rao theory are general enough to be applicable to a large class of models including ergodic and non-ergodic cases. This generalized Fisher-Rao model is referred to as the locally asymptotically mixed normal model (or LAMN model) (see Basawa-Scott (1983) [B-01] for the introduction to the LAMN model). We suppose that the MLE $\tilde{\theta}^{MLE}_{N,X^T_N}$ is a consistent estimator of θ_0, i.e.,

$$\tilde{\theta}^{MLE}_{N,X^T_N} \to \theta_0 \ \text{ in } \mathbb{Q}_{\theta_0} \text{ - probability as } N \to \infty, \tag{2.1.10}$$

and, under the LAMN model, the MLE $\tilde{\theta}^{MLE}_{N,X^T_N}$ is an asymptotically efficient estimator[5] of θ_0. In order to make an inference about θ_0 by the approximate MLE $\tilde{\theta}^{A}_{N,X^T_N,\Delta}$, we assume that $\tilde{\theta}^{MLE}_{N,X^T_N}$ and $\tilde{\theta}^{A}_{N,X^T_N,\Delta}$ share the same asymptotic distribution when N approaches infinity, and $\Delta(N)$, depending on N, approaches zero sufficiently fast. Namely, we have assumed that $\theta_0 \approx \tilde{\theta}^{A}_{N,X^T_N,\Delta}$ as $N \to \infty$ and $\Delta(N) \to 0^+$ sufficiently fast.

From Remark 2.1, we have assumed that when the number of observations N is large and the sampling interval Δ is sufficiently small, the true-parameter θ_0 can be approximated by the value of $\tilde{\theta}^{A}_{N,X^T_N,\Delta}$. Normally, futures prices are quoted almost everyday in a year (except on the holidays). This means we can approximate Δ by $1/365$. In a futures market of a commodity, many futures contracts of the commodity with different maturities are traded. To infer θ_0, we should start collecting the futures prices of the contract which has the longest lifetime because it gives a largest value of N. However, the lifetimes of the contracts are sometimes short, for example in AFET, the maximum lifetime is only 180 days for rice futures. This means the maximum value of N is 180 and we are not able to increase the number of observations. To overcome this problem, we suppose that there exists a futures contract of the commodity initiated at the first trading day of the futures market with a maturity date $T > 0$. From Equation (1.3.21), the logarithmic no-arbitrage futures price X^T_t can be written in terms of $F^{T_1}_t$ and $F^{T_2}_t$ as

$$X^T_t = \ln S\left(t, F^{T_1}_t, F^{T_2}_t; \theta\right) + A(T-t;\theta) + B(T-t;\theta)\,\delta(t, F^{T_1}_t, F^{T_2}_t;\theta), \tag{2.2.11}$$

for any $t \in [0,T]$, where $F^{T_1}_t$ and $F^{T_2}_t$ are two no-arbitrage futures prices on day t of the commodity with different maturities dates T_1 and T_2, respectively.

[5] See the definition of an asymptotically efficient estimator of a given true-parameter under LAMN model in Basawa-Scott (1983) [B-01].

Suppose that Assumption A holds. In other words, the no-arbitrage futures prices data defined by

$$F_N^T := \left\{ \left(F_{t_n}^{T_1^n}, F_{t_n}^{T_2^n} \right) : n = 1, \ldots, N \right\}, \tag{2.2.12}$$

for a positive integer N, can be observed from the futures market at the equidistant time points t_n. Hence, from Equation (2.2.2) and Equation (2.2.11), the approximate log-likelihood function of X_N^T can be written in terms of $F_{t_n}^{T_1^n}$ and $F_{t_n}^{T_2^n}, n = 1, \ldots, N,$ as

$$\begin{aligned} \tilde{l}_N^A(\theta; X_N^T(F_N^T), \Delta) = & -\frac{1}{2}\sum_{n=2}^{N} \ln\left(\pi\Delta\, \tilde{a}_T(t_n, F_{t_n}^{T_1^n}, F_{t_n}^{T_2^n}; t_{n-1}, F_{t_{n-1}}^{T_1^{n-1}}, F_{t_{n-1}}^{T_2^{n-1}}; \theta) \right) \\ & -\frac{1}{\Delta}\sum_{n=2}^{N} \frac{\left(X_{t_n}^T(F_{t_n}^{T_1^n}, F_{t_n}^{T_2^n}; \theta) - X_{t_{n-1}}^T(F_{t_{n-1}}^{T_1^{n-1}}, F_{t_{n-1}}^{T_2^{n-1}}; \theta) \right)^2}{\tilde{a}_T(t_n, F_{t_n}^{T_1^n}, F_{t_n}^{T_2^n}; t_{n-1}, F_{t_{n-1}}^{T_1^{n-1}}, F_{t_{n-1}}^{T_2^{n-1}}; \theta)}. \end{aligned} \tag{2.2.13}$$

By setting $t_1 = 0$ at the first trading day of the futures market and by setting today to be the maturity date or $t_N = T$, the no-arbitrage futures prices data F_N^T can be chosen from the futures market on every trading day from the first trading day up to today. This implies that the number of observations N is equal to the number of trading days of the futures market which is the maximum number of the observations that can be obtained from this futures market. In other words, we use the futures prices data from every trading day of the futures market to make an inference about the true-parameter θ_0.

In Chapter 3, we estimate the model (1.2.1) by applying Equation (2.2.13) to the rice futures prices data and the rubber futures prices data obtained from AFET. The two objective functions derived from the rice futures prices data and rubber futures prices data will be maximized to obtain the corresponding approximate MLEs of the log-futures prices process of rice and the log-futures prices process of rubber, respectively. The maximization will be done on a parameter set Θ under the following constraints:

$$\left. \begin{aligned} & p(\theta) > 0 \text{ and } |p_2(\theta)| < \sqrt{p(\theta)} \quad \text{(from Proposition 5)}, \\ & \left. \begin{aligned} & \beta_2 / \beta_1 \le \varepsilon_\beta \text{ for some } \varepsilon_\beta > 0, \\ & \left(\beta_1 \left(\alpha_0 - f_\alpha(T;\theta) \sum_{k=1}^{K^\alpha} \left| \alpha_k^{(1)} \right| + \left| \alpha_k^{(2)} \right| \right) + \kappa\beta_2 \right) / \sigma_\delta^2 \beta_1^2 \ge \frac{1}{2} \end{aligned} \right\} \text{conditions on } \delta_t, \\ & a_T(t_n, F_N^T; \theta) > 0, \quad n = 1, \ldots, N, \text{ (from Proposition 9)}, \end{aligned} \right\} \tag{2.2.14}$$

where $p(\theta)$ and $p_2(\theta)$ are given in Proposition 5. We employ the numerical procedure known as 'NMaximize[ ]' provided in *Mathematica* as a tool for solving the optimization problem (2.2.4) subject to the constraints (2.2.14).

Chapter 3

Applications to Agricultural Commodity Futures: The Cases of Rice and Natural Rubber in Thailand

The two main objectives of this chapter are to calibrate model (M) and to demonstrate the practical applicability of model (M) by using the daily futures prices data of the two agricultural commodities in Thailand: rice and natural rubber. This chapter commences with the introductory backgrounds in the productions and prices of rice and natural rubber in Thailand. Then we describe the futures prices data provided on the website of the AFET in terms of contract specifications and the sample time periods. Using the results obtained in the previous chapters, the parameters set and the constraints are specified for the estimation of the model parameters based on the maximum likelihood approach. The heuristic algorithm known as Differential Evolution (DE) provided in *Mathematica* is run to solve the optimization problems arising from the use of the estimation approach. The estimation results are reported with discussions focusing on the implications for the prices of rice and natural rubber in Thailand. Using the estimated parameters, we calculate price differences and correlations between the observed futures prices and their corresponding no-arbitrage (predicted) futures prices for several futures contracts in AFET of the two commodities. Finally, we analyze the implications of model (M) for capital budgeting decisions. We display the forward surfaces for the two commodities obtained from model (M) in the sample periods and a discussion about the situations known as backwardation and contango that can be observed on the forward surfaces is provided therein.

3.1 Rice and NR Productions and Prices in Thailand

Thailand has been one of the world major rice and natural rubber (NR) producers and exporters for more than 10 years. From the USDA online report 2006, Thailand has exported rice more than one forth of total world rice export since 2002. According to the Quarterly Natural Rubber Statistical Bulletin 2006 published by the Secretariat of the ANRPC, Thailand is the largest natural rubber exporter in the millennium by exporting more than two fifth of the total export in the ANRPC. Table 3.1 and Table 3.2, respectively, show the gross quantity of exports of rice and NR in the world major rice and NR export countries in the years 2002 -2004.

In Thailand, monsoon is generally characterized by distinct wet-season (during May through November) and dry-season (during December through April), which are associated with wide seasonal fluctuations in both rice and NR productions and prices. In the case of rice or "*Oryza sativa* " which is the vital food crop in the country, Thai farmers cultivate large amount of rice in wet-season because rice trunk needs abundant of water and moisture.

Table 3.1: Gross Quantity of Exports of Rice

Country \ Year	Thousand Metric Tons		
	2002/03	2003/04	2004/05
Argentina	170 (0.62)	249 (0.92)	345 (1.19)
Australia	141 (0.51)	131 (0.48)	52 (0.18)
Brazil	19 (0.07)	37 (0.14)	272 (0.94)
Burma, Union of	388 (1.41)	130 (0.48)	190 (0.65)
Cambodia	10 (0.04)	300 (1.10)	200 (0.69)
China, People Republic of	2,583 (9.37)	880 (3.24)	656 (2.26)
Egypt	579 (2.10)	826 (3.04)	1,095 (3.77)
EU-25	220 (0.80)	187 (0.69)	201 (0.69)
Guyana	200 (0.73)	243 (0.89)	182 (0.63)
India	4,421 (16.03)	3,172 (11.67)	4,687 (16.16)
Japan	200 (0.73)	200 (0.74)	200 (0.69)
Pakistan	1,958 (7.10)	1,986 (7.31)	3,032 (10.45)
Thailand	**7,552 (27.39)**	**10,137 (37.29)**	**7,274 (25.07)**
Uruguay	675 (2.45)	804 (2.96)	762 (2.63)
USA	3,834 (13.90)	3,090 (11.37)	3,862 (13.31)
Vietnam	3,795 (13.76)	4,295 (15.80)	5,174 (17.84)
Others	830 (3.01)	517 (1.90)	825 (2.84)
World Total	**27,575**	**27,184**	**29,009**

Source 1 : United States Department of Agriculture (USDA): Foreign Agriculture Service.

Note 01 : Numbers in parentheses indicate percentages of the corresponding World Total gross quantities of exports.

Note about date : 2002/03 is calendar year 2003, 2003/04 is calendar year 2004, and 2004/05 is calendar year 2005.

Table 3.2: Gross Quantity of Exports of NR from ANRPC countries

Country \ Year	Metric Tons*		
	2002	2003	2004
India	43,970 (0.87)	56,668 (1.03)	65,173 (1.09)
Indonesia	1,495,197 (29.51)	1,660,920 (30.32)	1,875,059 (31.46)
Malaysia	886,874 (17.50)	945,889 (17.27)	1,106,086 (18.56)
Papua New Guinea	4,207 (0.08)	4,092 (0.07)	4,495 (0.08)
Singapore	245,641 (4.85)	201,615 (3.68)	231,193 (3.88)
Sri Lanka	36,112 (0.71)	35,199 (0.64)	40,324 (0.68)
Thailand	**2,354,416 (46.47)**	**2,573,450 (46.98)**	**2,637,096 (44.25)**
ANRPC Total	**5,066,417**	**5,477,833**	**5,959,426**

Source 2 : the Quarterly Natural Rubber Statistical Bulletin 2006 published by the Secretariat of the ANRPC.

Note 02 : Numbers in parentheses indicate percentages of the corresponding ANRPC Total gross quantities of exports.

Note 03 : * includes domestic exports and re-exports.

However, the investments in irrigation schemes result a double cropping of rice in a crop year especially in the central plain and a modest expansion of upland crop cultivation in certain parts of the north and northeast. The increased rice output resulting from irrigation has increased the supply of rice available for domestic and foreign consumers. For rice, dry-season harvest (starts around the beginning of July) is mostly exported while wet-season harvest (starts around the end of November) is mainly supplied to the domestic market. The effect of the dry-season crop on seasonal rice price variations will depend partly on the size of the dry-season crop relative to that of the wet-season crop. If the dry-season crop is small, the large supply of the wet-season output that is placed on the market after the harvest will push prices down. In each month that follows, the price will increase by at least the cost of storage, until the wet-season harvest of the following year. When the large dry-season crop is marketed, this may dampen the seasonal price increase of the wet-season output, or may even cause prices to fall slightly.

In contrast to rice, NR is by product of latex from rubber tree "*Hevea brasiliensis*". It is a tropical tree which is native to the Amazon basin in Brazil and adjoining countries. *Hevea* was taken from the Amazon to Sri Lanka, Singapore, and Malaysia by the British Colonial Office where it was grown experimentally and later plantations. Subsequently, cultivation spread to Vietnam, Cambodia, Indonesia, and Thailand. *Hevea brasiliensis* grows best at temperatures of 20-28 °C with a well-distributed annual rainfall of 1,800-2,000 mm. *Hevea* trees convert inorganic nutrients from the soil, and carbon dioxide from the atmosphere, into organic carbohydrates which are then turned into rubber latex. At least once a year the leaves of trees, which are the sites of carbohydrate formation, die and fall off in wintering (in Thailand, around the end of November), and new leaves are formed. In wintering, which lasts for sixteen weeks, the metabolism of the tree and the constitution of its latex are substantially affected and the yield is reduced. Therefore, the prices of NR decline in that period. On the other hand, in wet-season, the yield is increased and the prices of NR are high since excessive rainfall interferes with tapping and collection of latex. This climatic factor accounts for marking seasonal variations in NR prices.

Figures 3.1 - 3.2, respectively, depict the evolutions of WR5 prices and RSRS3 prices in the years 2001-2003. It can be noticed from the graphs that, in each year, the graph exhibits two local minima and two local maxima in the case of rice. Meanwhile, in the case of NR, the graph exhibits one local minimum and one local maximum (except in 2003). As described above, these results come from the domestic demand-supply and the climatic factors of rice and NR. Consequently, in the estimation of the unknown parameters, we prefer $K^{\alpha} = 2$ in the case of rice and $K^{\alpha} = 1$ in the case of NR.

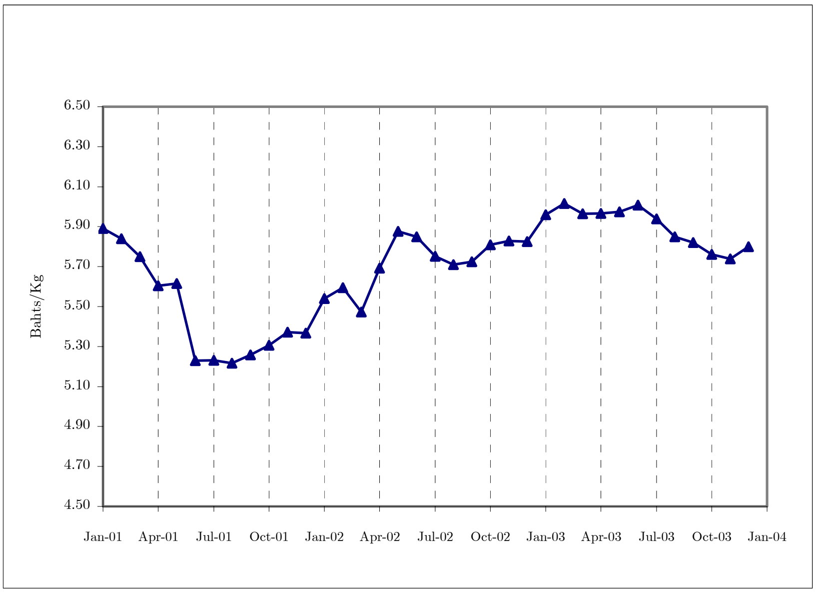


Figure 3.1: Evolution of WR5 prices in the years 2001-2003

Source 3 : Department of Internal Trade, Ministry of Commerce, Thailand: website: http://www.dit.go.th

Note about currency : "Baht" is the currency of Thailand.

Note 04 : The prices are the averaged prices in month of White Rice 5%. (WR5).

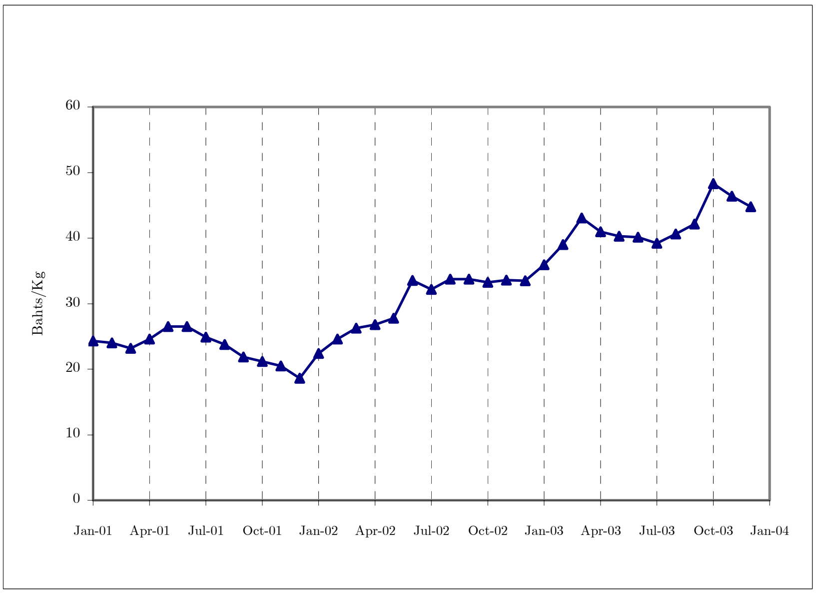


Figure 3.2: Evolution of RSRS3 prices in the years 2001-2003

Source 4 : Office of the Rubber Aid Fund, Thailand: website: http://www.rubber.co.th

Note 05 : The prices are the averaged prices in month of Ribbed Smoked Rubber Sheet No. 3 (RSRS3).

Besides the domestic demand-supply and the climatic factors, the other two factors that influence rice and NR prices in Thailand are the Thai price intervention scheme and the world prices of rice and NR. Thai government operates the intervention scheme with the objectives of stabilizing and supporting the domestic prices received by rice farmers and NR smallholders. This price support scheme has apparently achieved its objectives, but at a very high cost, partly borne by the government and partly by the domestic processing industry. World prices of rice and NR have also influenced to domestic prices because the small domestic usage of the commodities causes the domestic prices are determined by the world demand. One can incorporate the two factors into the model by adding a jump-diffusion term for the former factor and/or introducing a diffusion process describing the world prices for the latter factor. This will be an interesting work. The influences of these two factors on rice and NR prices are, however, beyond this research's scope.

3.2 Rice and Rubber Futures Prices Data

Rice futures and rubber futures have been traded in the Agricultural Futures Exchange of Thailand (AFET) since the middle of the year 2004. AFET considers White Rice 5% (WR5) and Ribbed Smoked Rubber Sheet no.3 (RSRS3) as the underlying commodities for the rice futures and the rubber futures, respectively. WR5 is the largest quantity in Thai rice market since WR5 is typical rice than other kinds of rice. Moreover, WR5 which is the middle grade can be upgraded to White Rice 100% or downgraded to White Rice 10%. Therefore, the prices of WR5 determine the prices of the other kinds. For rubber, RSRS3 is preferred because it is easily transport, storage, and its standard is globally accepted.

The data used to estimate the model parameters consist of daily observations of futures prices for two agricultural commodities, WR5 and RSRS3, obtained from AFET. The sample period for WR5 futures is the 26/08/2004 through 26/08/2005 (approximately one year) and the sample period for RSRS3 futures is the 26/08/2004 through 25/08/2006 (approximately two years). In fact, the daily data of WR5 futures are available after the sample period. But the futures prices were quite stable due to the small trading volume in each trading day and hence we do not prefer them. Unlike WR5 futures, the RSRS3 futures trading volumes were high throughout the sample period and then we use the futures prices data of RSRS3 until August 2006. To summarize the data, WR5 futures and RSRS3 futures being available in the sample periods are tabulated in Table 3.3.

Table 3.3: WR5 futures contracts traded in AFET in the period 26/08/2004 - 26/08/2005 and RSRS3 futures contracts traded in AFET in the period 26/08/2004 - 26/08/2006

Month	Futures Contracts	
	WR5 -	RSRS3 -
Aug 04	**NOV04**, **DEC04**, JAN05, MAR05, MAY05, JUL05	**SEP04**, **OCT04**, NOV04, DEC04, JAN05, FEB05, MAR05
Sep 04	**NOV04**, **DEC04**, JAN05, MAR05, MAY05, JUL05, SEP05	**OCT04**, **NOV04**, DEC04, JAN05, FEB05, MAR 05, APR05
Oct 04	**NOV04**, **DEC04**, JAN05, MAR05, MAY05, JUL05, SEP05	**NOV04**, **DEC04**, JAN05, FEB05, MAR05, APR05, MAY05
Nov 04	**NOV04**, **DEC04**, **JAN05**, MAR05, MAY05, JUL05, SEP05	**DEC04**, **JAN05**, FEB05, MAR05, APR05, MAY05, JUN05
Dec 04	**DEC04**, **JAN05**, **MAR05**, MAY05, JUL05, SEP05	**JAN05**, **FEB05**, MAR05, APR05, MAY05, JUN05, JUL05
Jan 05	**JAN05**, **MAR05**, **MAY05**, JUN05, JUL05, SEP05	**FEB05**, **MAR05**, APR05, MAY05, JUN05, JUL05, AUG05
Feb 05	**MAR05**, **APR05**, MAY05, JUN05, JUL05, SEP05	**MAR05**, **APR05**, MAY05, JUN05, JUL05, AUG05, SEP05
Mar 05	**MAR05**, **APR05**, **MAY05**, JUN05, JUL05, AUG05, SEP05	**APR05**, **MAY05**, JUN05, JUL05, AUG05, SEP05, OCT05
Apr 05	**APR05**, **MAY05**, **JUN05**, JUL05, AUG05, SEP05, OCT05	**MAY05**, **JUN05**, JUL05, AUG05, SEP05, OCT05, NOV05
May 05	**MAY05**, **JUN05**, **JUL05**, AUG05, SEP05,OCT05, NOV05	**JUN05**, **JUL05**, AUG05, SEP05, OCT05, NOV05, DEC05
Jun 05	**JUN05**, **JUL05**, **AUG05**, SEP05,OCT05, NOV05, DEC05	**JUL05**, **AUG05**, SEP05, OCT05, NOV05, DEC05, JAN06
Jul 05	**JUL05, AUG05**, **SEP05**, OCT05, NOV05, DEC05, JAN06	**AUG05**, **SEP05**, OCT05, NOV05, DEC05, JAN06, FEB06
Aug 05	**AUG05**, **SEP05**, **OCT05**, NOV05, DEC05, JAN06, FEB06	**SEP05**, **OCT05**, NOV05, DEC05, JAN06, FEB06, MAR06
Sep 05	-	**OCT05**, **NOV05**, DEC05, JAN06, FEB06, MAR06, APR06
Oct 05	-	**NOV05**, **DEC05**, JAN06, FEB06, MAR06, APR06, MAY06
Nov 05	-	**DEC05**, **JAN06**, FEB06, MAR06, APR06, MAY06, JUN06
Dec 05	-	**JAN06**, **FEB06**, MAR06, APR06, MAY06, JUN06, JUL06
Jan 06	-	**FEB06**, **MAR06**, APR06, MAY06, JUN06, JUL06, AUG06
Feb 06	-	**MAR06**, **APR06**, MAY06, JUN06, JUL06, AUG06, SEP06
Mar 06	-	**APR06**, **MAY06**, JUN06, JUL06, AUG06, SEP06, OCT06
Apr 06	-	**MAY06**, **JUN06**, JUL06, AUG06, SEP06, OCT06, NOV06
May 06	-	**JUN06**, **JUL06**, AUG06, SEP06, OCT06, NOV06, DEC06
Jun 06	-	**JUL06**, **AUG06**, SEP06, OCT06, NOV06, DEC06, JAN07
Jul 06	-	**AUG06**, **SEP06**, OCT06, NOV06, DEC06, JAN07, FEB07
Aug 06	-	**SEP06**, **OCT06**, NOV06, DEC06, JAN07, FEB07, MAR07

Note 06 : We abbreviate the WR5 futures contract matured at the fifth of November 2004 by WR5 -NOV04, and so on.

Note 07 : We abbreviate the RSRS3 futures contract matured at the first of September 2004 by RSRS3 -SEP04, and so on.

Note 08 : The futures contracts underlined and having boldfaces in each month are used as data for estimation of the parameters in the model.

From Assumption A, in each trading day, the futures prices data from the closest maturity contract and the second-closest maturity contract are sufficient for estimation of the parameters. As illustrated in Table 3.3, in rice case, the prices data of WR5–NOV04 and WR5-DEC04 are used from August 2004 to November 2004. Since every futures contract of WR5 expires on the fifth the corresponding maturity month, the prices data of WR5-

JAN05 are used after the contract WR-NOV04 expired. Unlike the futures contracts of WR5, every RSRS3 futures contract expires on the first of the corresponding maturity month. Therefore, in each month, only two futures contracts of RSRS3 are used. In our implementation, we set the date 26/08/2004 to be the starting date of the model, i.e., $t_1 = 0$ at the date. For WR5, we consider the futures contract of WR5 with maturity date $T_W = 1$ and, for RSRS3, we consider the futures contract of RSRS3 with maturity date $T_R = 2$. We then denote the set of the no-arbitrage futures prices data of WR5 by $F^{T_W}_{N_W}$, i.e., $F^{T_W}_{N_W} = \left\{\left(F^{T^n_{1W}}_{t_n}, F^{T^n_{2W}}_{t_n}\right) : n = 1,\dots,N_W\right\}$. Likewise, we denote the set of the no-arbitrage futures prices data of RSRS3 by $F^{T_R}_{N_R}$, i.e., $F^{T_R}_{N_R} = \left\{\left(F^{T^n_{1R}}_{t_n}, F^{T^n_{2R}}_{t_n}\right) : n = 1,\dots,N_R\right\}$, where N_W and N_R are, respectively, the numbers of observations of the no-arbitrage WR5 and RSRS3 futures prices. The data sets, $F^{T_W}_{N_W}$ and $F^{T_R}_{N_R}$, are demonstrated in Figure 3.3.

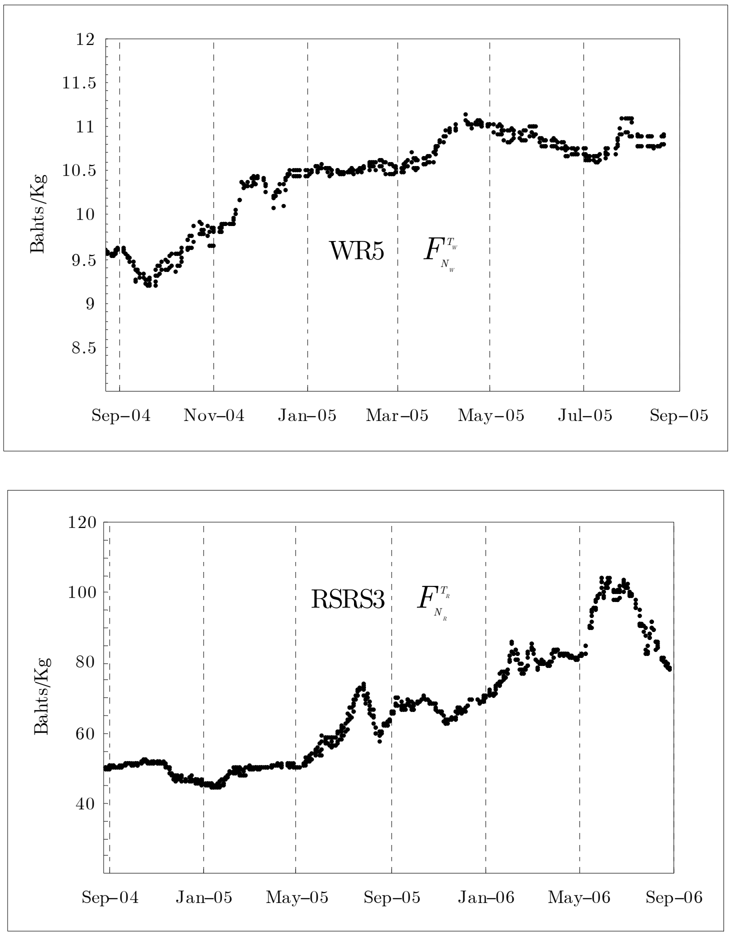


Figure 3.3: Futures prices data of WR5 and RSRS3 obtained from AFET: http://www.afet.or.th

3.3 The Parameters Set and the Constraints

We apply the futures prices data $F^{T_W}_{N_W}$ and $F^{T_R}_{N_R}$ into Equation (2.2.13). Then we obtain the approximate log-likelihood functions of log-futures prices data of WR5 and RSRS3, denoted by $\tilde{l}^{A}_{N_W}(\theta; F^{T_W}_{N_W}, \Delta)$ and $\tilde{l}^{A}_{N_R}(\theta; F^{T_R}_{N_R}, \Delta)$, respectively. Suppose that the risk free interest rates are constant in the sample periods and the true-parameters of WR5 and RSRS3 log-futures prices processes and their corresponding MLEs, denoted by $\theta^W_0, \theta^R_0, \theta^W_{MLE}$, and θ^R_{MLE}, respectively, are interior points of the parameter set Θ descried as follows:

$$\varepsilon_1 \le \beta_1, \beta_2 \le \varepsilon_2, \tag{3.3.1}$$

$$\varepsilon_1 \le \sigma_\delta, \kappa \le \varepsilon_3, \tag{3.3.2}$$

$$-\varepsilon_4 \le \lambda_S, \lambda_\delta, \alpha_0, \alpha^{(1)}_k, \alpha^{(2)}_k \le \varepsilon_4,\ k = 1, \ldots, K^\alpha, \tag{3.3.3}$$

$$0 \le \rho \le 1, \tag{3.3.4}$$

for some $\varepsilon_i > 0, i = 1, \ldots, 4$. We solve the optimization problems:

$$\tilde{\theta}^{A,W}_{N_W,\Delta} = \arg\max_{\theta \in \Theta} \tilde{l}^{A}_{N_W}(\theta; F^{T_W}_{N_W}, \Delta) \text{ for WR5,} \tag{3.3.5}$$

and

$$\tilde{\theta}^{A,R}_{N_R,\Delta} = \arg\max_{\theta \in \Theta} \tilde{l}^{A}_{N_R}(\theta; F^{T_R}_{N_R}, \Delta) \text{ for RSRS3,} \tag{3.3.6}$$

under the constraints (from the constraints (2.2.14)):

$$p(\theta) \ge \varepsilon_1 \text{ and } |p_2(\theta)| + \varepsilon_1 \le \sqrt{p(\theta)} \tag{3.3.7}$$

$$\left.\begin{array}{c} \beta_2 / \beta_1 \le 1, \\ \left(\beta_1\left(\alpha_0 - f_\alpha(T;\theta)\sum_{k=1}^{K^\alpha}\left|\alpha^{(1)}_k\right| + \left|\alpha^{(2)}_k\right|\right) + \kappa\beta_2\right) / \sigma^2_\delta \beta^2_1 \ge \dfrac{1}{2}, \end{array}\right\} \text{conditions on } \delta_t, \tag{3.3.8}$$

$$a_T(t_n, F^{T_W}_{N_W}; \theta) \ge \varepsilon_1, \quad n = 1, \ldots, N_W \quad \text{for WR5,} \tag{3.3.9}$$

$$a_T(t_n, F^{T_R}_{N_R}; \theta) \ge \varepsilon_1, \quad n = 1, \ldots, N_R \quad \text{for RSRS3,} \tag{3.3.10}$$

where $K^\alpha = 2$ for WR5 and $K^\alpha = 1$ for RSRS3, $p(\theta), p_2(\theta)$ are given in Proposition 5.

We suppose further that θ^W_0, θ^R_0, θ^W_{MLE}, and θ^R_{MLE} admit the above corresponding constraints. Hence, under the assumptions imposed in Remark 2.1, we have $\theta^W_0 \approx \tilde{\theta}^{A,W}_{N_W,\Delta}$ and $\theta^R_0 \approx \tilde{\theta}^{A,R}_{N_R,\Delta}$ when N_W, N_R are large and Δ is sufficiently small.

According to the statistical data proposed by the Bank of Thailand available on the website: http://www.bot.or.th, interest rates fluctuated between 3.5% and 5% over the sample periods. Thus, in our estimation, we use a risk free interest rate of 5% which is the maximum rate in the sample period. To avoid the possibility of over-level convenience yields, all unknown parameters (except ρ) are set for having value in the specific ranges described in Expressions (3.3.1)-(3.3.3) by setting $\varepsilon_1 = 10^{-5}, \varepsilon_2 = \varepsilon_4 = 1,$ and $\varepsilon_3 = 2.$ The correlation coefficient between the commodity spot price and its volatility (or ρ) is considered only in the non-negative range since we are interested in the *inverse leverage effect* as explained in Subsection 1.1.1 of Chapter 1. Finally, we set the ratio β_2 / β_1 to be bounded by one which is the maximum rate of carrying yields as described in Subsection 1.2.2 of Chapter 1.

3.4 Heuristic Algorithm for the Optimization Problems

The optimization problems derived in the previous section are complex problems and they become more complex as the numbers of observations increase. Although exact optimization algorithms such as Quadratic Programming and Sequential Quadratic Programming, can be applied to solve the problems, it may take a large amount of calculation time to reach the maximizers. Therefore, we prefer heuristic algorithms which are faster and easy to implement than the exact algorithms. There are several heuristic algorithms can be the methods of choice. Such those algorithms, we choose Differential Evolution (DE) (see Storn(2005) [S-03]). Besides its good convergence properties, DE is very simple to understand and to implement. DE is a parallel direct search method which utilizes *NP* vectors in a *d*-dimensional parameter space, i.e., $\theta_{i,G} \in \mathbb{R}^d, i = 1,\ldots, NP-1,$ as a population for each generation *G*, i.e., for each iteration of the maximization. For fixed *NP*, the initial population is chosen randomly and should try to cover the entire parameter space uniformly. Basically, DE generates new parameter vectors by adding the weighted difference between two population vectors to a third vector. If the resulting vector yields a higher objective function value than a predetermined population member, the newly generated vector replaces the vector, at which it was compared, in the next generation; otherwise, the old vector is retained. *Mathematica* provides DE as an option of the built-in functions known as 'NMaximize[ ]' and we employ them to solve the optimization problems.

3.5 Estimation Results and Discussions

In this section we estimate the model (1.2.1) based on the futures prices data $F_{N_w}^{T_w}$ and $F_{N_R}^{T_R}$. The estimation results for WR5 and RSRS3 obtained by solving the optimization problems (3.3.5)-(3.3.6) in Section 3.3 are tabulated in Table 3.4.

Table 3.4: Parameter estimates for WR5 futures contracts sample: 26/08/2004 - 26/08/2005 and Parameter estimates for RSRS3 futures contracts sample: 26/08/2004 - 26/08/2006

Parameter	Approximate MLE for WR5 (White Rice 5%)	Approximate MLE for RSRS3 (Ribbed Smoked Rubber Sheet no. 3)
β_1	**0.0509** (±0.0027) [0.0455, 0.0563]	**0.6332** (± 0.1291) [0.3750, 0.8914]
β_2	**0.0474** (± 0.0022) [0.0430, 0.0518]	**0.5564** (± 0.1322) [0.2920, 0.8208]
κ	**0.3752** (± 0.0666) [0.2420, 0.5084]	**1.9916** (± 0.0733) [1.8450, 2.1382]
σ_δ	**0.0146** (± 0.0012) [0.0112, 0.0170]	**1.6311** (± 0.2816) [1.0679, 2.1943]
λ_S	**- 0.2014** (± 0.2064) [-0.6142, 0.2114]	**- 0.0372** (± 0.0364) [-0.1100, 0.0356]
λ_δ	**- 0.5532** (± 0.2040) [-0.9612, -0.1452]	**- 0.0100** (± 0.0030) [-0.0160, -0.0040]
ρ	**0.6073** (± 0.0248) [0.5577, 0.6569]	**0.1223** (± 0.0190) [0.0843, 0.1603]
α_0	**0.4240** (± 0.0735) [0.2770, 0.5710]	**- 0.0984** (± 0.0052) [-0.1088, -0.0880]
$\alpha_1^{(1)}$	**0.2431** (± 0.0165) [0.2101, 0.2761]	**- 0.0741** (± 0.0038) [-0.0817, -0.0665]
$\alpha_2^{(1)}$	**- 0.0039** (± 0.0002) [-0.0043, -0.0035]	**- 0.0999** (± 0.0140) [-0.1279, -0.0719]
$\alpha_1^{(2)}$	**- 0.2188** (± 0.0928) [-0.4044, -0.0332]	-
$\alpha_2^{(2)}$	**- 0.3772** (± 0.0970) [-0.5712, -0.1832]	-
K^α	2	1
No. of observations	245	471
Log-likelihood value	747.045	812.45

Note 09 : The numbers having boldfaces are the parameter estimates.

Note 10 : The numbers in the parentheses are the approximate asymptotic standard deviations of the MLE s obtained by the nonparametric bootstrap procedures.

Note 11 : The closed intervals are of the form [x - 2s, x + 2s] where x is the estimate and s is the standard deviation.

Note 12 : Throughout the following discussion we will refer to an estimate being significantly different from a given value if the given value is not within the corresponding closed interval [x -2s, x+2s] (the usually applied asymptotic 95% confidence interval). For example, the 95% confidence interval of β_1 is [0.0455, 0.0563] for WR5.

The following discussion of the full-sample parameter estimates in Table 3.4 starts by focusing on the estimates of the parameters that describe the dynamics of the model (2.1.1). Subsequently, using these estimated parameters, we extract spot prices, convenience yields, price volatilities, and seasonality of the two commodities in Subsections 3.5.1-3.5.2. In Subsection 3.5.3, we provide two main discussions about the spot prices and the convenience yields. Firstly, we discuss about the relationship between the extracted spot prices and the daily observed spot prices in Bangkok of the two commodities. Secondly, we discuss about the relation between convenience yields, price volatilities and seasonality by focusing on the implications for the prices of the two commodities in Thailand.

We first consider the parameters β_1 and β_2 which are of the particular interest because they describe the impact of convenience yields on the volatilities of both spot prices and convenience yields. In addition, the exclusion of either would yield a significant simplification of the model (1.2.1). Looking at the point estimates in Table 3.4, the estimates of β_1 are of different magnitudes (0.05 for WR5 and 0.60 for RSRS3) as same as the estimates of β_2 (0.03 for WR5 and 0.56 for RSRS3). Moreover, they are significantly different from zero. Similarly, the estimates of σ_δ, describing the impact of convenience yields on volatility of themselves, and the estimates of ρ, the correlation between the commodity spot prices and the instantaneous convenience yields, are significantly different from zero at the 5% level with different magnitudes.

The estimates of κ and α_0 which are significantly different from zero justify the use of the mean-reverting process to model the convenience yield process. Likewise, the estimates of the seasonal parameters $\alpha_k^{(1)}$ and $\alpha_k^{(2)}, k = 1, 2,$ are significantly different from zero at the 5% level. The inclusion of the seasonal component in the convenience yield process is the core feature of the continuous-time modeling in this research. In particular, the two-factor model in Nielsen-Schwartz (2004) [N-02] is basically the special case where the seasonal parameters are equal to zero.

The estimates of λ_δ, the convenience yield risk, are significantly different from zero while, the estimates of λ_S, the market prices of risk, have large standard deviations and they are not significantly different from zero in both commodities. This implies that no strong inference can be drawn from the estimates of λ_S. Since we will use the estimated parameters to extract spot prices, convenience yields, and futures prices by using the extraction formulas (1.3.19)-(1.3.21), we provide an investigation of the sensitivities of the parameter λ_S to the extraction formulas in Appendix H. We compute the first derivatives of the functions with respect to λ_S at every observed time point. If such the first derivatives are small, the variations of λ_S within the confident intervals will not affect much the extracted values of the spot prices, the convenience yields, and the futures prices.

3.5.1 Extractions of Commodity Prices and Convenience Yields

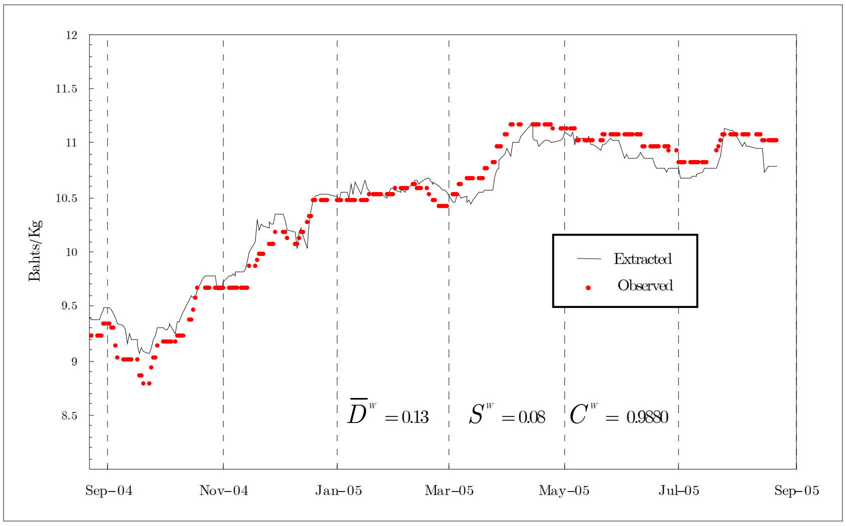
In order to extract the spot prices and the instantaneous convenience yields of WR5 and RSRS3 in the corresponding sample periods, we apply the futures data $F_{N_w}^{T_w}$ and $F_{N_R}^{T_R}$, respectively, into the extraction formulas (1.3.19)-(1.3.20) by setting the parameters to be equal to the approximate MLEs tabulated in Table 3.4 - 3.5. In the case of WR5, we plot the extracted spot prices versus the daily observed market spot prices of WR5 in Bangkok. The graph is illustrated in Figure 3.5 together with the average of the absolute price differences between the extracted spot prices and the observed spot prices (denoted by $\bar{D}^W$), the standard deviation of the absolute price differences (denoted by S^W) and the Pearson's correlation efficient between the extracted spot prices and the observed spot prices (denoted by C^W). The extracted instantaneous convenience yields of WR5 are shown in Figure 3.6. In the similar way of WR5, for the case of RSRS3, we plot the extracted spot prices of RSRS3 versus the daily observed market spot prices of RSRS3 in Bangkok. The graph is illustrated in Figure 3.7 together with $\bar{D}^R, S^R$, and C^R. The extracted instantaneous convenience yields of RSRS3 are shown in Figure 3.8.

3.5.2 Extractions of Price Volatilities and Seasonality

The daily volatilities of the returns (or the daily price volatilities) on WR5 spot prices and RSRS3 spot prices in the sample periods can be obtained by taking square root to the term $\beta_1\delta_{t_n} + \beta_2$ where δ_{t_n} is the corresponding instantaneous convenience yield on day t_n. The seasonal functions of WR5 and RSRS3 are transformed by using the transformation:

$$\alpha_T(t) \mapsto \tilde{\alpha}_T(t) := \left(\alpha_T(t) - (\alpha_0 + \varepsilon_1^\alpha)\right)\varepsilon_2^\alpha,$$

for some $\varepsilon_i^\alpha > 0, i = 1, 2,$ to have suitable ranges which can be displayed together with the corresponding volatilities and the instantaneous convenience yields. We note here that the transformation $\tilde{\alpha}_T(t)$ preserves the local maxima and the local minima of $\alpha_T(t)$. For WR5, the daily price volatilities and the transformed seasonal function are illustrated in Figure 3.6 and, for RSRS3, the daily price volatilities and the transformed seasonal function are illustrated in Figure 3.8.



Source 5 : The daily observed spot prices of WR5 are obtained from Department of Internal Trade, Ministry of Commerce, Thailand.

Figure 3.5: The extracted spot prices and the observed spot prices of WR5

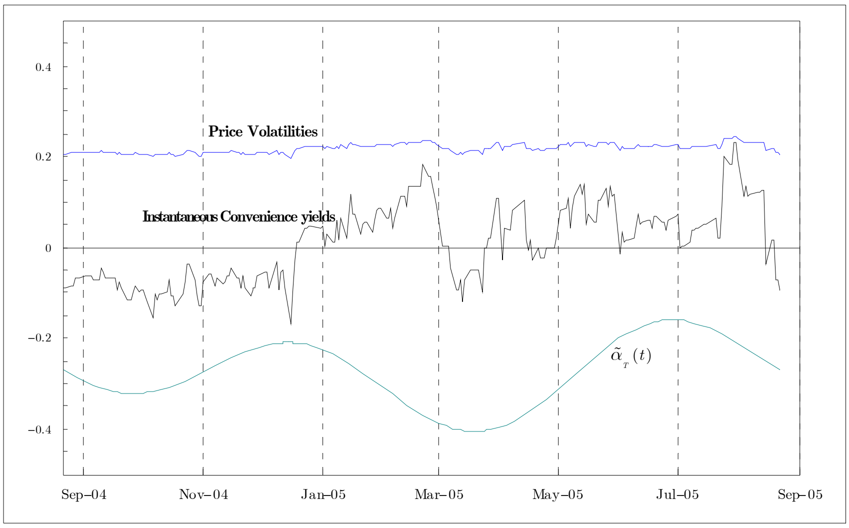
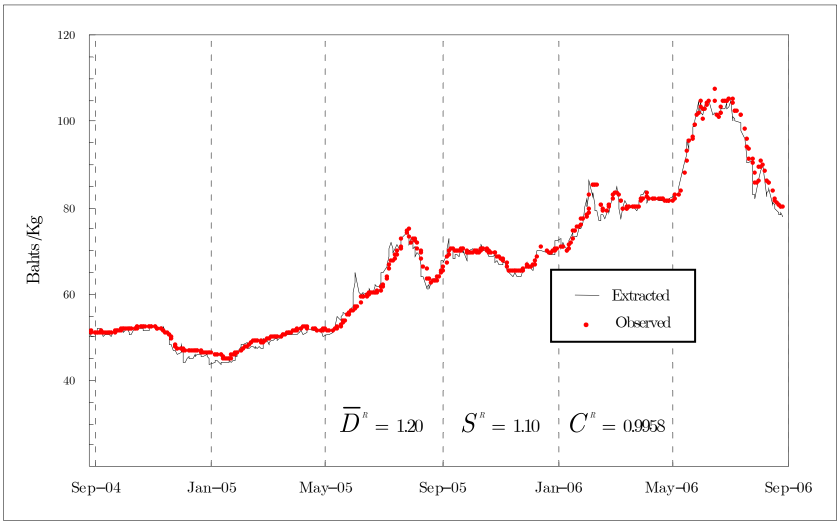


Figure 3.6: The convenience yields versus volatilities and seasonality in WR5 prices

Note 13 : The average of the instantaneous convenience yields of WR5 in the period Sep-04 to Aug-05 is 0.0074 and the standard deviation is 0.0846.

Note 14 : We investigate the sensitivities of the parameter λ_s to the extracted values of the spot prices and the convenience yields of WR5 in Appendix H. The results obtained show that the average of the prediction errors for the extracted spot prices in the sample period is ± 0.001 Bahts/kg and the average of the prediction errors for the extracted convenience yields in the sample period is ± 0.022.



Source 6 : The daily observed spot prices of RSRS3 are obtained from the office of the Rubber Aid Fund, Thailand.

Figure 3.7: The extracted spot prices and the observed spot prices of RSRS3

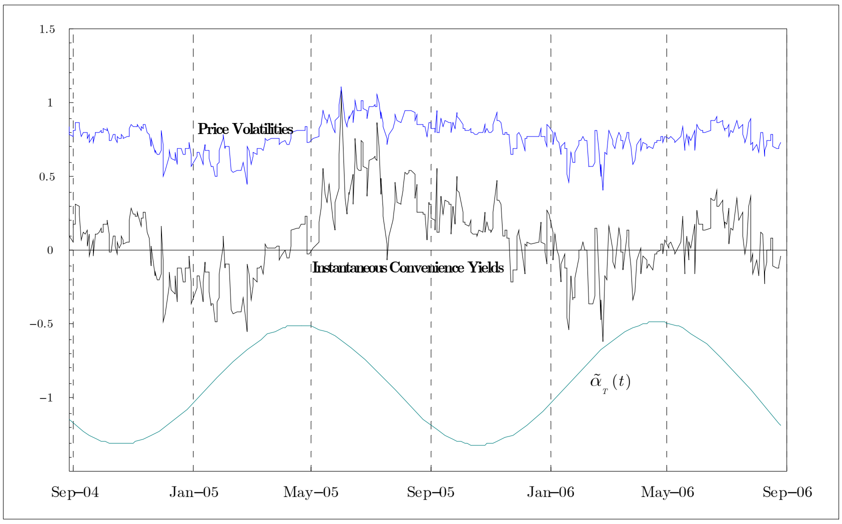


Figure 3.8: The convenience yields versus volatilities and seasonality in RSRS3 prices

Note 15 : The average of the instantaneous convenience yields of RSRS3 in the period Sep-04 to Aug-05 is 0.0917 and the standard deviation is 0.3024.

Note 16 : The average of the instantaneous convenience yields of RSRS3 in the period Sep-05 to Aug-06 is 0.0602 and the standard deviation is 0.1972.

Note 17 : The average of the instantaneous convenience yields of RSRS3 in the period Sep-04 to Aug-06 is 0.0759 and the standard deviation is 0.2553.

Note 18 : We investigate the sensitivities of the parameter λ_S to the extracted values of the spot prices and the convenience yields of RSRS3 in Appendix H. The results obtained show that the average of the prediction errors for the extracted spot prices in the sample period is ± 0.06 Bahts/kg and the average of the prediction errors for the extracted convenience yields in the sample period is ± 0.051.

3.5.3 Discussions

Daily Observed Spot Prices vs Extracted Spot Prices

As displayed in Figure 3.5, the extracted spot prices of WR5 are close and strongly correlated to the daily observed market spot prices in Bangkok with the average of the absolute price differences is 0.13 Baht/kg, the standard deviation is 0.08 Baht/kg, and the correlation is 0.988. These results indicate that the spot prices of WR5 in Bangkok had a strong influence to the traders in the futures market of WR5. In other words, the traders considered the spot prices of WR5 in Bangkok as the underlying prices before preferring the futures prices of WR5 in AFET. The same situation also happened in the futures market of RSRS3, as seen from Figure 3.7, the average absolute price difference is 1.20 Bahts/kg, the standard deviation is 1.10 Bahts/kg, and the correlation is 0.9958.

Convenience Yields vs Volatilities and Seasonality in WR5 Case

We first consider the convenience yields of WR5 in the sample period. As shown in Figure 3.6, the convenience yields mostly exhibited positive in the periods December through March and April through August. These results can be explained by the seasonal function $\alpha_T(t)$ effecting the convenience yield process. It can be seen from the graph that $\tilde{\alpha}_T(t)$ has two local maxima and two local minima. The local maximum is achieved in December and the global maximum is achieved in July. As described in Section 3.1, wet-season harvest of rice started around the beginning of December and dry-season harvest of rice started around the beginning of July. Thus, the convenience yields and the spot prices of WR5 were high in the two harvesting time periods. Meanwhile, the local minimum is achieved in October and the global minimum is achieved around the end of March. Consequently, the convenience yields were negative from the beginning of the sample period through the middle of December and they became negative again around the middle of March. These results reflect that the inventories of WR5 were high and the spot prices of WR5 declined after harvesting. It should be noticed here that the high level of convenience yields in August, was the influence of the Thai government price support scheme since the prices of WR5 usually decline after the dry-season harvesting and the supply of WR5 is plenty in that month. The daily volatilities of the returns on WR5 spot prices fluctuated in the narrow range 0.22 ± 0.03 with the average is 0.22, the standard deviation is 0.01, the maximum is 0.24323, and the minimum is 0.196817.

Convenience Yields vs Volatilities and Seasonality in RSRS3 Case

The convenience yields of RSRS3 in the sample period depicted in Figure 3.8 mostly exhibited positive in the periods September 2004 to October 2004, April 2005 to October 2005, and May 2006 to June 2006. These results can be explained in a similar way as in the case of WR5. As described in Section 3.1, in wet-season, the convenience yields of RSRS3 increase and RSRS3 prices are high since excessive rainfall interferes with tapping and collecting of latex. On the other hand, in the periods December 2004 to March 2005 and January 2006 to April 2006, the convenience yields mostly exhibited negative. This implies the inventories of RSRS3 were high and the yields reduced in the dry-season periods. The just obtained results reveal the seasonal variation in RSRS3 prices which can be determined by $\alpha_T(t)$. As shown by the graph of $\tilde{\alpha}_T(t)$ in Figure 3.8, in each year of the sample period, the global minimum is achieved in October about two months prior to the beginning of dry-season and the global maxima is achieved in April about two months prior to the beginning of wet-season. We now return to the convenience yields of RSRS3 as shown in Figure 3.8. It should be pointed out here that the average of the convenience yields for the first year of the sample period are higher than the average of the convenience yields for the second year of the sample period. This is because of the world demand of RSRS3 was very high while the world supply of RSRS3 was limited in the years 2004 and 2005. China, for example, imported large amount of RSRS3 from Thailand and other ANRPC countries to consume the domestic demand. However, the world demand of RSRS3 declined in 2006 and the yields were low. The daily volatilities of the returns on RSRS3 spot prices fluctuated substantially in the range 0.77±0.40 with the average is 0.77, the standard deviation is 0.11, the maximum is 1.1129, and the minimum is 0.4012.

WR5 vs RSRS3 in Price Volatilities

It can be noticed from Figures 3.6 and 3.8 that WR5 price volatilities were lower than RSRS3 price volatilities in the period Sep-04 to Sep-05. Moreover, the daily WR5 prices volatilities were less variable than the daily RSRS3 price volatilities. This is likely to be a result of the Thai price intervention scheme operated in rice prices, which tends to stabilize the domestic rice prices at a specific level. This intervention scheme was also operated in the domestic NR prices, but with a smaller size compare to the rice case. This is because, in Thailand, an increasing in rice prices affects more consumers than an increasing in NR prices.

3.6 Observed vs Predicted Futures Prices in AFET

In this section, we consider price differences and correlations between the observed futures prices and their corresponding no-arbitrage futures prices of the two commodities, rice and natural rubber. The observed futures prices are obtained from several futures contracts of WR5 and RSRS3 that had been traded in AFET in the sample periods. Plugging the estimated parameters of WR5 and RSRS3 tabulated in Table 3.4 into the extraction formula (1.3.21), we then obtain the corresponding no-arbitrage futures prices of WR5 and RSRS3, respectively. Throughout the following discussions we will refer to the predicted futures prices as the corresponding no-arbitrage futures prices. We start the first subsection by introducing measurements for price differences and correlations. The two subsequent subsections report the price differences and the price correlations via two series of graphs. In the last subsection, we discuss about the results obtained from this consideration.

3.6.1 Measurements for Price Differences and Correlations

For a futures contract of commodity **F** traded in AFET with a maturity date $\overline{T}$, we denote $n_{\mathbf{F}}^{\overline{T}}$ is number of days in which the futures contract had been traded in the sample periods. Then we can express $n_{\mathbf{F}}^{\overline{T}}$ as follows:

$$n_{\mathbf{F}}^{\overline{T}} = n_{\mathbf{F}}^{\overline{T},1} + n_{\mathbf{F}}^{\overline{T},2}, \tag{3.6.1}$$

where $n_{\mathbf{F}}^{\overline{T},1}$ is the number of days in which we do not use the futures prices of the contract as the observed data in estimation of the model parameters and $n_{\mathbf{F}}^{\overline{T},2}$ is the number of the days in which we use the futures prices of the contract as the observed data.

Let $0 \le t_1 < t_2 < \dots < t_{n_{\mathbf{F}}^{\overline{T}}} \le \overline{T}$ be the trading days of the futures contract. From Assumption A, the trading days can be separated into two sequences as follows:

$$P_{\mathbf{F}}^{\overline{T},1} := \left\{t_1, t_2, \dots, t_{n_{\mathbf{F}}^{\overline{T},1}}\right\} \quad \text{and} \quad P_{\mathbf{F}}^{\overline{T},2} := \left\{t_{n_{\mathbf{F}}^{\overline{T},1}+1}, t_{n_{\mathbf{F}}^{\overline{T}}+2}, \dots, t_{n_{\mathbf{F}}^{\overline{T}}}\right\},$$

where $\left|P_{\mathbf{F}}^{T,1}\right| = n_{\mathbf{F}}^{T,1}$ and $\left|P_{\mathbf{F}}^{T,2}\right| = n_{\mathbf{F}}^{T,2}$. In order to calculate the no-arbitrage futures price on day t_i, we use the extraction formula (1.3.21), the estimated parameters, and the sample data $\left(F_{t_i}^{T_1^i}, F_{t_i}^{T_2^i}\right)$ from the no-arbitrage futures prices data F_N^T. Applying this procedure to every point in $P_{\mathbf{F}}^{\overline{T},1}$ and $P_{\mathbf{F}}^{\overline{T},2}$ gives us the following two sequences of the no-arbitrage futures prices:

$$FP_{\mathbf{F}}^{\bar{T},1} := \left(F_{\mathbf{F}}^{\bar{T},1}(t_1), F_{\mathbf{F}}^{\bar{T},1}(t_2), \ldots, F_{\mathbf{F}}^{\bar{T},1}\left(t_{n_{\mathbf{F}}^{\bar{T}}}\right)\right)$$

and
$$FP_{\mathbf{F}}^{\bar{T},2} := \left(F_{\mathbf{F}}^{\bar{T},2}\left(t_{n_{\mathbf{F}}^{\bar{T}}+1}\right), F_{\mathbf{F}}^{\bar{T},2}\left(t_{n_{\mathbf{F}}^{\bar{T}}+2}\right), \ldots, F_{\mathbf{F}}^{\bar{T},2}\left(t_{n_{\mathbf{F}}^{\bar{T}}}\right)\right).$$

Similarly, the daily observed futures prices of the futures contract at every point in $P_{\mathbf{F}}^{\bar{T},1}$ and $P_{\mathbf{F}}^{\bar{T},2}$ can be represented as the following two sequences:

$$OP_{\mathbf{F}}^{\bar{T},1} := \left(O_{\mathbf{F}}^{\bar{T},1}(t_1), O_{\mathbf{F}}^{\bar{T},1}(t_2), \ldots, O_{\mathbf{F}}^{\bar{T},1}\left(t_{n_{\mathbf{F}}^{\bar{T}}}\right)\right)$$

and
$$OP_{\mathbf{F}}^{\bar{T},2} := \left(O_{\mathbf{F}}^{\bar{T},2}\left(t_{n_{\mathbf{F}}^{\bar{T}}+1}\right), O_{\mathbf{F}}^{\bar{T},2}\left(t_{n_{\mathbf{F}}^{\bar{T}}+2}\right), \ldots, O_{\mathbf{F}}^{\bar{T},2}\left(t_{n_{\mathbf{F}}^{\bar{T}}}\right)\right).$$

Under Assumption A, we have $FP_{\mathbf{F}}^{\bar{T},2} = OP_{\mathbf{F}}^{\bar{T},2}$. Moreover, if AFET is arbitrage-free, we must have $FP_{\mathbf{F}}^{\bar{T},1} = OP_{\mathbf{F}}^{\bar{T},1}$. However, $FP_{\mathbf{F}}^{\bar{T},1}$ and $OP_{\mathbf{F}}^{\bar{T},1}$ are sometimes different. Therefore, we want to investigate the differences and the correlations between the two sequences for several futures contracts of WR5 and RSRS3 that had been traded in AFET in the sample periods in order to demonstrate the practical applicability of our model. Namely, the model is applicable for the two commodity prices if, for each selected futures contract, the two sequences are not significantly different and strongly positive correlated.

In our investigation, we measure the differences between the observed futures prices and their corresponding no-arbitrage (predicted) futures prices for a futures contract of the commodity **F** using the average of percentage absolute price differences between $FP_{\mathbf{F}}^{\bar{T},1}$ and $OP_{\mathbf{F}}^{\bar{T},1}$ defined by

$$\bar{D}_{\mathbf{F}}^{\bar{T}} := \frac{1}{n_{\mathbf{F}}^{\bar{T},1}} \sum_{i=1}^{n_{\mathbf{F}}^{\bar{T},1}} D_{\mathbf{F}}^{\bar{T},i}, \tag{3.6.2}$$

where

$$D_{\mathbf{F}}^{\bar{T},i} := \left| \frac{FP_{\mathbf{F}}^{\bar{T},1}(t_i) - OF_{\mathbf{F}}^{\bar{T},1}(t_i)}{FP_{\mathbf{F}}^{\bar{T},1}(t_i)} \right| \times 100\%, \tag{3.6.3}$$

is the percentage absolute prices difference between $FP_{\mathbf{F}}^{\bar{T},1}$ and $OP_{\mathbf{F}}^{\bar{T},1}$ on day t_i. It should be noted that this measurement has no influence from the dimension of the futures prices. Namely, $\bar{D}_{\mathbf{F}}^{\bar{T}}$ does not depend on the currency unit of the futures prices. One can see that the two sequences $FP_{\mathbf{F}}^{\bar{T},1}$ and $OP_{\mathbf{F}}^{\bar{T},1}$ coincide if and only if $\bar{D}_{\mathbf{F}}^{\bar{T}} = 0$. Furthermore, we use the sample standard deviation of the percentage absolute price difference between $FP_{\mathbf{F}}^{\bar{T},1}$ and $OP_{\mathbf{F}}^{\bar{T},1}$ defined by

$$S_{\mathbf{F}}^{\bar{T}} := \sqrt{\frac{1}{n_{\mathbf{F}}^{\bar{T},1} - 1} \sum_{i=1}^{n_{\mathbf{F}}^{\bar{T},1}} \left(D_{\mathbf{F}}^{\bar{T},i} - \bar{D}_{\mathbf{F}}^{\bar{T}} \right)^2}, \tag{3.6.4}$$

to measure the variability in the sample $D_{\mathbf{F}}^{\bar{T},1}, D_{\mathbf{F}}^{\bar{T},2}, \ldots, D_{\mathbf{F}}^{\bar{T},n_{\mathbf{F}}^{\bar{T},1}}$. Finally, we employ Pearson's correlation coefficient between $FP_{\mathbf{F}}^{\bar{T},1}$ and $OP_{\mathbf{F}}^{\bar{T},1}$ defined by

$$C_{\mathbf{F}}^{\bar{T}} := \frac{\sum_{i=1}^{n_{\mathbf{F}}^{\bar{T},1}} (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n_{\mathbf{F}}^{\bar{T},1}} (x_i - \bar{x})^2} \sqrt{\sum_{i=1}^{n_{\mathbf{F}}^{\bar{T},1}} (y_i - \bar{y})^2}}, \tag{3.6.5}$$

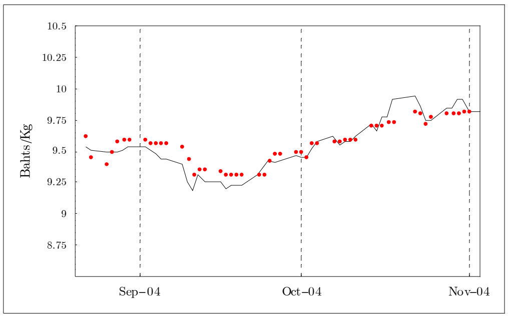
to measure the strength of the relationship between $FP_{\mathbf{F}}^{\bar{T},1}$ and $OP_{\mathbf{F}}^{\bar{T},1}$, where x_i, y_i are the i^{th} elements of the two sequences $FP_{\mathbf{F}}^{\bar{T},1}$ and $OP_{\mathbf{F}}^{\bar{T},1}$, respectively, and $\bar{x}, \bar{y}$ are the arithmetic means of the two sequences $FP_{\mathbf{F}}^{\bar{T},1}$ and $OP_{\mathbf{F}}^{\bar{T},1}$, respectively.

3.6.2 Observed vs Predicted Futures Prices of WR5 and RSRS3

In this subsection, we employ the measurements introduced in the previous subsection to investigate price differences and correlations between observed futures prices of WR5 and predicted futures prices of WR5. After that we investigate in RSRS3 case. We consider for several futures contracts of WR5 and RSRS3 that had been traded in the sample periods: 26/08/2004 - 26/08/2005 for WR5 and 26/08/2004 - 26/08/2006 for RSRS3. For WR5 case, we select 6 futures contracts: WR5-MAR05, WR5-MAY05, WR-JUN05, WR-JUL05, WR5-AUG05, and WR5-SEP05. For RSRS3 case, we select 24 futures contracts in which their maturity dates belong to the period: 01/12/2004 - 01/11/2006. For each selected futures contract, we calculate $\bar{D}_{\mathbf{F}}^{\bar{T}}$, $S_{\mathbf{F}}^{\bar{T}}$, and $C_{\mathbf{F}}^{\bar{T}}$. The results obtained are demonstrated via two series of graphs. In each graph, we plot the elements of the sequence $FP_{\mathbf{F}}^{\bar{T},1}$ versus the elements of the sequence $OP_{\mathbf{F}}^{\bar{T},1}$ evolving in time. In addition, we consider price differences and correlations on a time period close to the maturity date $\bar{T}$. We recalculate $\bar{D}_{\mathbf{F}}^{\bar{T}}$, $S_{\mathbf{F}}^{\bar{T}}$, and $C_{\mathbf{F}}^{\bar{T}}$ based on the 10 trading days before $t_{n_{\mathbf{F}}^{\bar{T},1}+1}$ and we show the new values in the parentheses under the previous values. The discussion about the results obtained from this subsection is provided in the next subsection.

Observed vs Predicted Futures Prices of WR5

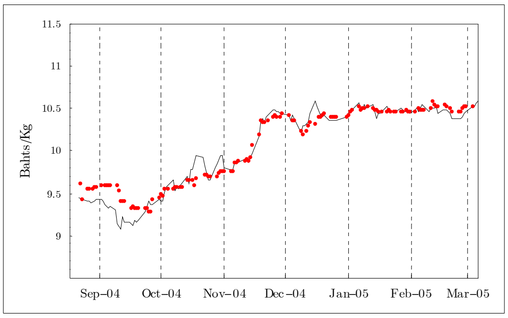
WR5-MAR05



F = **WR5** $\bar{T}$ = MAR05 $n_F^{\bar{T}} = 131$ $n_F^{\bar{T},1} = 50$ $n_F^{\bar{T},2} = 81$

$\bar{D}_F^{\bar{T}} = 0.70793\,\%$ (0.74354 %) $S_F^{\bar{T}} = 0.46237\,\%$ (0.48910 %) $C_F^{\bar{T}} = 0.95169$ (0.58481)

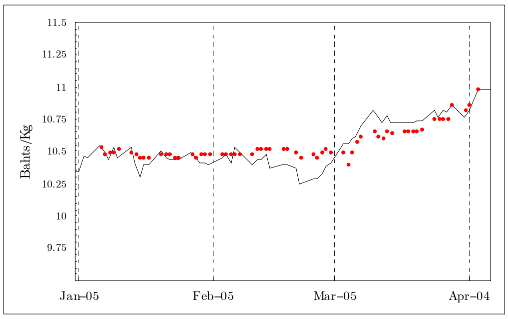
WR5-MAY05



F = **WR5** $\bar{T}$ = MAY05 $n_F^{\bar{T}} = 169$ $n_F^{\bar{T},1} = 131$ $n_F^{\bar{T},2} = 38$

$\bar{D}_F^{\bar{T}} = 0.82962\,\%$ (0.79550 %) $S_F^{\bar{T}} = 0.85596\,\%$ (0.17730 %) $C_F^{\bar{T}} = 0.97568$ (0.92581)

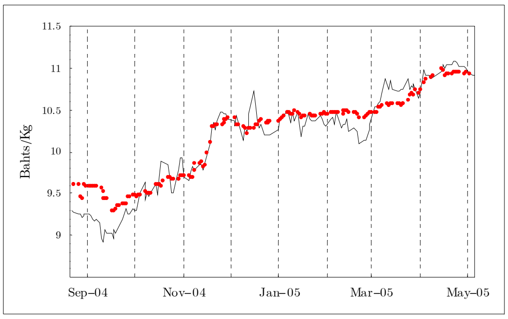
WR5-JUN05



F = **WR5** $\bar{T}$ = JUN05 $n_F^{\bar{T}} = 97$ $n_F^{\bar{T},1} = 61$ $n_F^{\bar{T},2} = 36$

$\bar{D}_F^{\bar{T}} = 0.70960\,\%$ (0.43459 %) $S_F^{\bar{T}} = 0.49971\,\%$ (0.25026 %) $C_F^{\bar{T}} = 0.86785$ (0.77934)

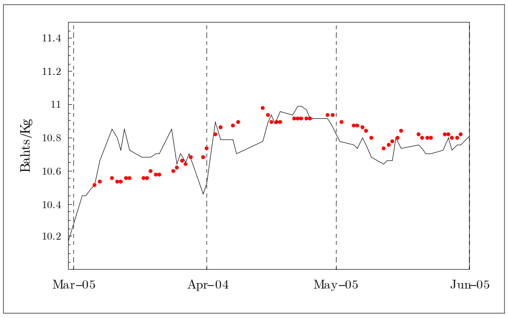
WR5-JUL05



F = **WR5** $\bar{T}$ = JUL05 $n_F^{\bar{T}} = 210$ $n_F^{\bar{T},1} = 169$ $n_F^{\bar{T},2} = 41$

$\bar{D}_F^{\bar{T}} = 1.45767\,\%$ (0.71892 %) $S_F^{\bar{T}} = 1.28413\,\%$ (0.25084 %) $C_F^{\bar{T}} = 0.96453$ (0.25320)

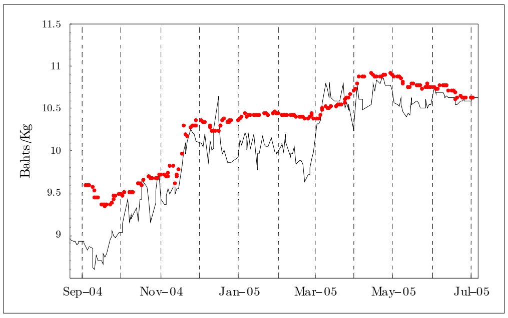
WR5-AUG05



F = **WR5** $\bar{T}$ = AUG05 $n_F^{\bar{T}} = 97$ $n_F^{\bar{T},1} = 55$ $n_F^{\bar{T},2} = 42$

$\bar{D}_F^{\bar{T}} = 0.92599\,\%$ (0.57521 %) $S_F^{\bar{T}} = 0.68562\,\%$ (0.25482 %) $C_F^{\bar{T}} = 0.54142$ (0.40590)

WR5-SEP05



F = **WR5** $\bar{T}$ = SEP05 $n_F^{\bar{T}} = 236$ $n_F^{\bar{T},1} = 201$ $n_F^{\bar{T},2} = 35$

$\bar{D}_F^{\bar{T}} = 3.00390\,\%$ (0.53733 %) $S_F^{\bar{T}} = 2.23330\,\%$ (0.17755 %) $C_F^{\bar{T}} = 0.93725$ (0.80204)

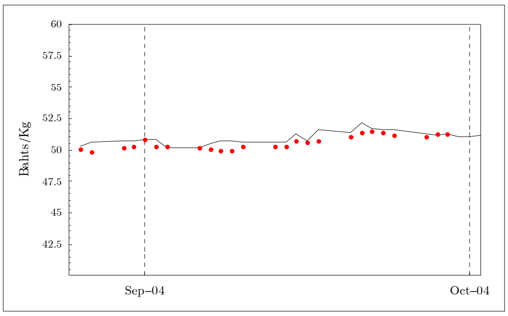
— Predicted price

• Observed price

Note : The number in the parenthesis under the value is obtained by recalculating the value based on the 10 trading days before $t_{n_F^{\bar{T},1}+1}$.

Observed vs Predicted Futures Prices of RSRS3

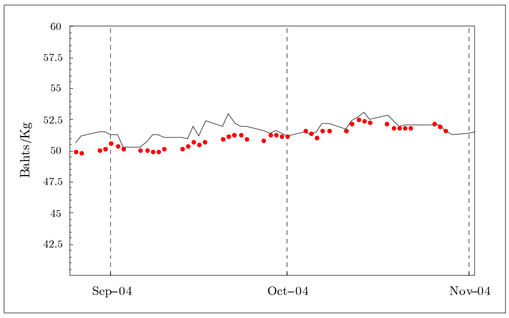
RSRS3-DEC04



$\mathbf{F} = \mathbf{RSRS3}$ $\quad \overline{T} = \mathrm{DEC04}$ $\quad n_{\mathbf{F}}^{\overline{T}} = 66$ $\quad n_{\mathbf{F}}^{\overline{T},1} = 25$ $\quad n_{\mathbf{F}}^{\overline{T},2} = 41$

$\overline{D}_{\mathbf{F}}^{\overline{T}} = 0.80415\,\%$ (0.77263 %) $\quad S_{\mathbf{F}}^{\overline{T}} = 0.51891\,\%$ (0.51982 %) $\quad C_{\mathbf{F}}^{\overline{T}} = 0.84550$ (0.63050)

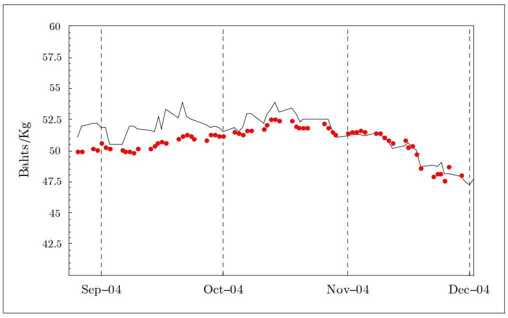
RSRS3-JAN05



$\mathbf{F} = \mathbf{RSRS3}$ $\quad \overline{T} = \mathrm{JAN05}$ $\quad n_{\mathbf{F}}^{\overline{T}} = 86$ $\quad n_{\mathbf{F}}^{\overline{T},1} = 44$ $\quad n_{\mathbf{F}}^{\overline{T},2} = 42$

$\overline{D}_{\mathbf{F}}^{\overline{T}} = 1.25441\,\%$ (0.65953 %) $\quad S_{\mathbf{F}}^{\overline{T}} = 0.93809\,\%$ (0.44050 %) $\quad C_{\mathbf{F}}^{\overline{T}} = 0.77733$ (0.73892)

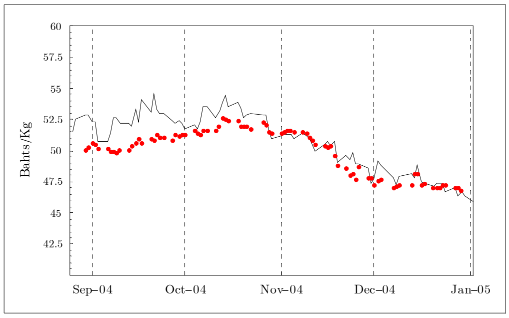
RSRS3-FEB05



$\mathbf{F} = \mathbf{RSRS3}$ $\quad \overline{T} = \mathrm{FEB05}$ $\quad n_{\mathbf{F}}^{\overline{T}} = 106$ $\quad n_{\mathbf{F}}^{\overline{T},1} = 66$ $\quad n_{\mathbf{F}}^{\overline{T},2} = 40$

$\overline{D}_{\mathbf{F}}^{\overline{T}} = 1.67435\,\%$ (0.95704 %) $\quad S_{\mathbf{F}}^{\overline{T}} = 1.30410\,\%$ (0.56252 %) $\quad C_{\mathbf{F}}^{\overline{T}} = 0.79061$ (0.90880)

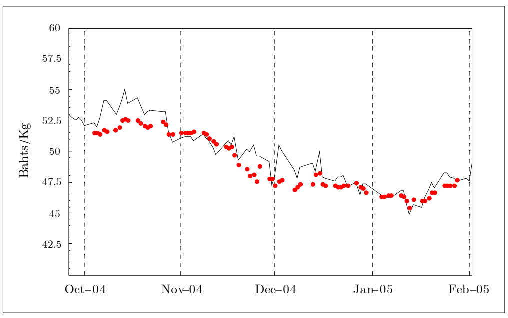
RSRS3-MAR05



$\mathbf{F} = \mathbf{RSRS3}$ $\quad \overline{T} = \mathrm{MAR05}$ $\quad n_{\mathbf{F}}^{\overline{T}} = 122$ $\quad n_{\mathbf{F}}^{\overline{T},1} = 84$ $\quad n_{\mathbf{F}}^{\overline{T},2} = 38$

$\overline{D}_{\mathbf{F}}^{\overline{T}} = 2.00327\,\%$ (0.66870 %) $\quad S_{\mathbf{F}}^{\overline{T}} = 1.58036\,\%$ (0.53352 %) $\quad C_{\mathbf{F}}^{\overline{T}} = 0.90048$ (0.82672)

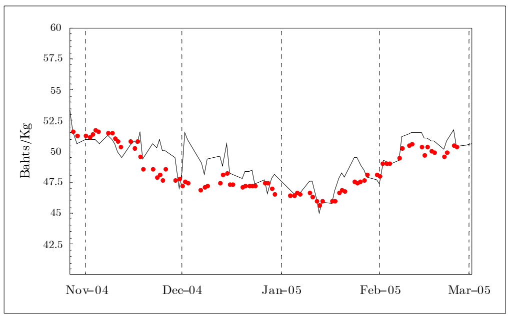
RSRS3-APR05



$\mathbf{F} = \mathbf{RSRS3}$ $\quad \overline{T} = \mathrm{APR05}$ $\quad n_{\mathbf{F}}^{\overline{T}} = 119$ $\quad n_{\mathbf{F}}^{\overline{T},1} = 79$ $\quad n_{\mathbf{F}}^{\overline{T},2} = 40$

$\overline{D}_{\mathbf{F}}^{\overline{T}} = 1.75418\,\%$ (1.32826 %) $\quad S_{\mathbf{F}}^{\overline{T}} = 1.29907\,\%$ (0.55283 %) $\quad C_{\mathbf{F}}^{\overline{T}} = 0.93987$ (0.93348)

RSRS3-MAY05



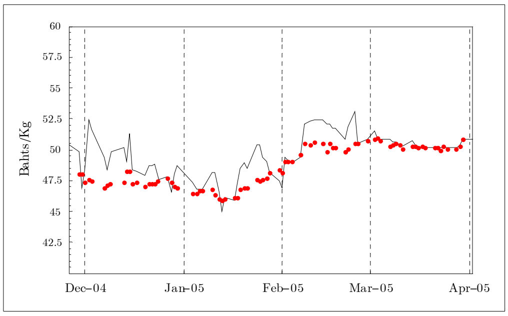
$\mathbf{F} = \mathbf{RSRS3}$ $\quad \overline{T} = \mathrm{MAY05}$ $\quad n_{\mathbf{F}}^{\overline{T}} = 119$ $\quad n_{\mathbf{F}}^{\overline{T},1} = 80$ $\quad n_{\mathbf{F}}^{\overline{T},2} = 39$

$\overline{D}_{\mathbf{F}}^{\overline{T}} = 1.97327\,\%$ (1.80150 %) $\quad S_{\mathbf{F}}^{\overline{T}} = 1.56596\,\%$ (0.49730 %) $\quad C_{\mathbf{F}}^{\overline{T}} = 0.81732$ (0.83055)

Predicted price
Observed price

Note : The number in the parenthesis under the value is obtained by recalculating the value based on the 10 trading days before $t_{n_{\mathbf{F}}^{\overline{T},1}+1}$.

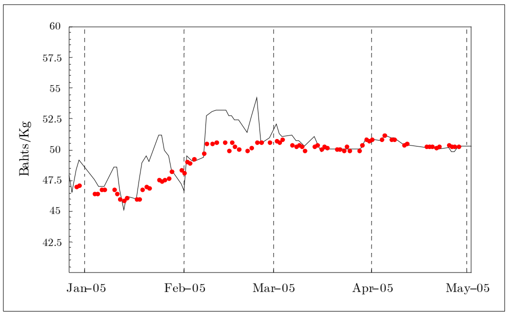
RSRS3-JUN05



$\mathbf{F} = \mathbf{RSRS3}$ $\quad \overline{T} = \mathrm{JUN05}$ $\quad n_{\mathrm{F}}^{\overline{T}} = 114$ $\quad n_{\mathrm{F}}^{\overline{T},1} = 80$ $\quad n_{\mathrm{F}}^{\overline{T},2} = 34$

$\overline{D}_{\mathrm{F}}^{\overline{T}} = 2.13319\,\%$ (0.19262 %) $\quad S_{\mathrm{F}}^{\overline{T}} = 1.98411\,\%$ (0.12608 %) $\quad C_{\mathrm{F}}^{\overline{T}} = 0.77818$ (0.27053)

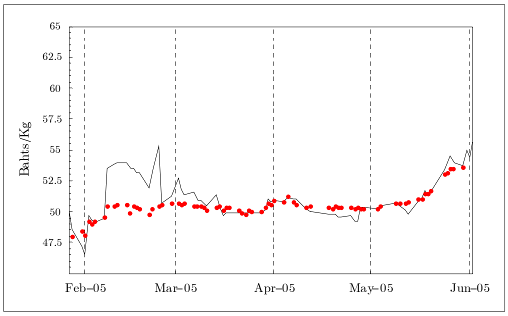
RSRS3-JUL05



$\mathbf{F} = \mathbf{RSRS3}$ $\quad \overline{T} = \mathrm{JUL05}$ $\quad n_{\mathrm{F}}^{\overline{T}} = 115$ $\quad n_{\mathrm{F}}^{\overline{T},1} = 77$ $\quad n_{\mathrm{F}}^{\overline{T},2} = 38$

$\overline{D}_{\mathrm{F}}^{\overline{T}} = 1.85691\,\%$ (0.37782 %) $\quad S_{\mathrm{F}}^{\overline{T}} = 2.01195\,\%$ (0.34910 %) $\quad C_{\mathrm{F}}^{\overline{T}} = 0.76899$ (0.59924)

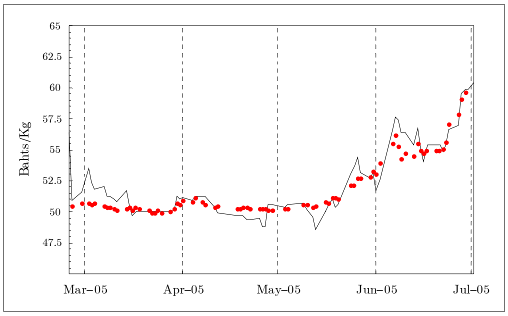
RSRS3-AUG05



$\mathbf{F} = \mathbf{RSRS3}$ $\quad \overline{T} = \mathrm{AUG05}$ $\quad n_{\mathrm{F}}^{\overline{T}} = 113$ $\quad n_{\mathrm{F}}^{\overline{T},1} = 74$ $\quad n_{\mathrm{F}}^{\overline{T},2} = 39$

$\overline{D}_{\mathrm{F}}^{\overline{T}} = 1.65741\,\%$ (0.86517 %) $\quad S_{\mathrm{F}}^{\overline{T}} = 2.04109\,\%$ (0.69532 %) $\quad C_{\mathrm{F}}^{\overline{T}} = 0.63006$ (0.97954)

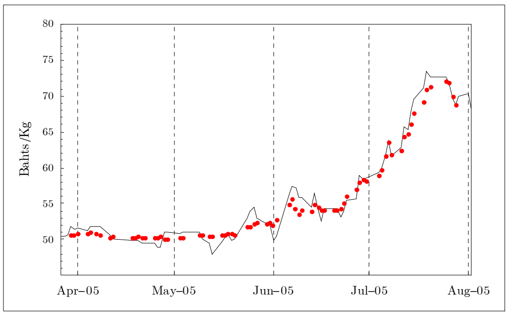
RSRS3-SEP05



$\mathbf{F} = \mathbf{RSRS3}$ $\quad \overline{T} = \mathrm{SEP05}$ $\quad n_{\mathrm{F}}^{\overline{T}} = 117$ $\quad n_{\mathrm{F}}^{\overline{T},1} = 77$ $\quad n_{\mathrm{F}}^{\overline{T},2} = 40$

$\overline{D}_{\mathrm{F}}^{\overline{T}} = 1.37083\,\%$ (0.77661 %) $\quad S_{\mathrm{F}}^{\overline{T}} = 1.11205\,\%$ (0.43594 %) $\quad C_{\mathrm{F}}^{\overline{T}} = 0.93603$ (0.94390)

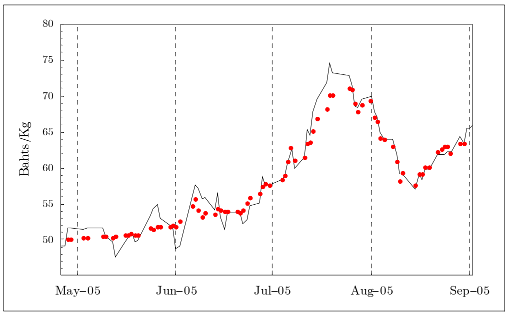
RSRS3-OCT05



$\mathbf{F} = \mathbf{RSRS3}$ $\quad \overline{T} = \mathrm{OCT05}$ $\quad n_{\mathrm{F}}^{\overline{T}} = 117$ $\quad n_{\mathrm{F}}^{\overline{T},1} = 73$ $\quad n_{\mathrm{F}}^{\overline{T},2} = 44$

$\overline{D}_{\mathrm{F}}^{\overline{T}} = 1.71782\,\%$ (1.76444 %) $\quad S_{\mathrm{F}}^{\overline{T}} = 1.25541\,\%$ (1.06081 %) $\quad C_{\mathrm{F}}^{\overline{T}} = 0.98717$ (0.95423)

RSRS3-NOV05



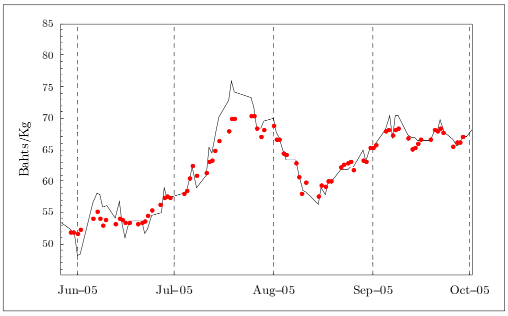
$\mathbf{F} = \mathbf{RSRS3}$ $\quad \overline{T} = \mathrm{NOV05}$ $\quad n_{\mathrm{F}}^{\overline{T}} = 120$ $\quad n_{\mathrm{F}}^{\overline{T},1} = 78$ $\quad n_{\mathrm{F}}^{\overline{T},2} = 42$

$\overline{D}_{\mathrm{F}}^{\overline{T}} = 2.04662\,\%$ (0.95060 %) $\quad S_{\mathrm{F}}^{\overline{T}} = 1.80958\,\%$ (0.64415 %) $\quad C_{\mathrm{F}}^{\overline{T}} = 0.97666$ (0.93869)

Predicted price
Observed price

Note : The number in the parenthesis under the value is obtained by recalculating the value based on the 10 trading days before $t_{n_{\mathrm{F}}^{\overline{T},1}+1}$.

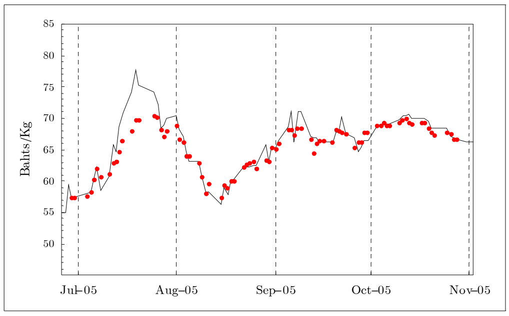
RSRS3-DEC05



F $=$ **RSRS3** $\quad \overline{T} =$ DEC05 $\quad n_{F}^{\overline{T}} = 125 \quad n_{F}^{\overline{T},1} = 83 \quad n_{F}^{\overline{T},2} = 42$

$\overline{D}_{F}^{\overline{T}} = 2.02681\,\%$ (0.69289 %) $\quad S_{F}^{\overline{T}} = 2.02520\,\%$ (0.69924 %) $\quad C_{F}^{\overline{T}} = 0.96843$ (0.84698)

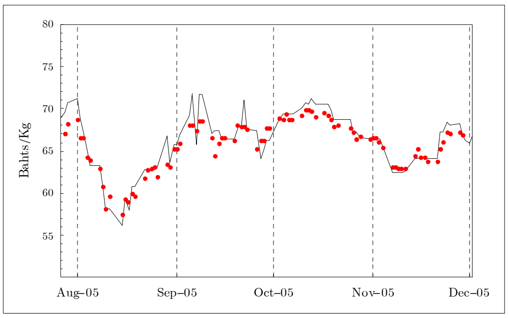
RSRS3-JAN06



F $=$ **RSRS3** $\quad \overline{T} =$ JAN06 $\quad n_{F}^{\overline{T}} = 119 \quad n_{F}^{\overline{T},1} = 82 \quad n_{F}^{\overline{T},2} = 37$

$\overline{D}_{F}^{\overline{T}} = 1.72592\,\%$ (1.15612 %) $\quad S_{F}^{\overline{T}} = 1.96193\,\%$ (0.503878 %) $\quad C_{F}^{\overline{T}} = 0.93830$ (0.94782)

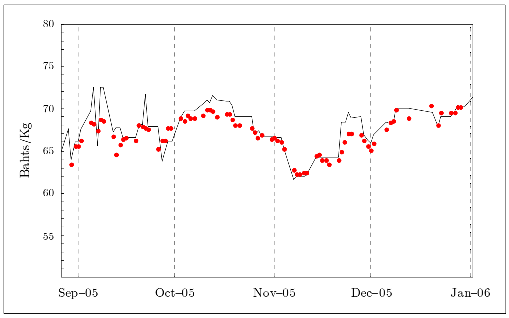
RSRS3-FEB06



F $=$ **RSRS3** $\quad \overline{T} =$ FEB06 $\quad n_{F}^{\overline{T}} = 121 \quad n_{F}^{\overline{T},1} = 86 \quad n_{F}^{\overline{T},2} = 35$

$\overline{D}_{F}^{\overline{T}} = 1.43748\,\%$ (1.22704 %) $\quad S_{F}^{\overline{T}} = 1.27542\,\%$ (0.92068 %) $\quad C_{F}^{\overline{T}} = 0.94578$ (0.92068)

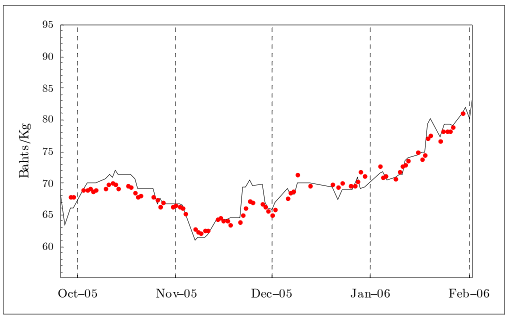
RSRS3-MAR06



F $=$ **RSRS3** $\quad \overline{T} =$ MAR06 $\quad n_{F}^{\overline{T}} = 117 \quad n_{F}^{\overline{T},1} = 79 \quad n_{F}^{\overline{T},2} = 38$

$\overline{D}_{F}^{\overline{T}} = 1.56631\,\%$ (0.61119 %) $\quad S_{F}^{\overline{T}} = 1.44209\,\%$ (0.49470 %) $\quad C_{F}^{\overline{T}} = 0.87029$ (0.77064)

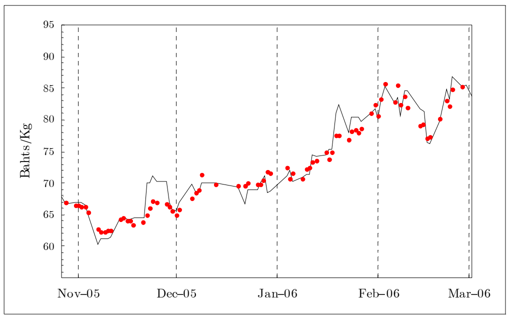
RSRS3-APR06



F $=$ **RSRS3** $\quad \overline{T} =$ APR06 $\quad n_{F}^{\overline{T}} = 117 \quad n_{F}^{\overline{T},1} = 77 \quad n_{F}^{\overline{T},2} = 40$

$\overline{D}_{F}^{\overline{T}} = 1.62097\,\%$ (1.44338 %) $\quad S_{F}^{\overline{T}} = 1.24076\,\%$ (1.01362 %) $\quad C_{F}^{\overline{T}} = 0.95215$ (0.91284)

RSRS3-MAY06



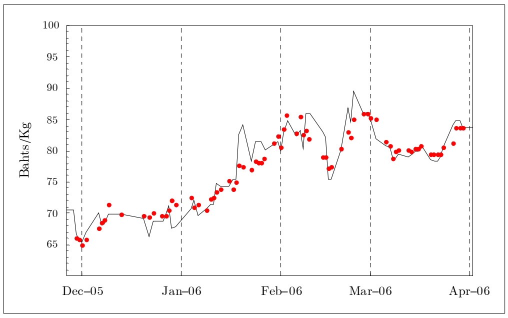
F $=$ **RSRS3** $\quad \overline{T} =$ MAY06 $\quad n_{F}^{\overline{T}} = 113 \quad n_{F}^{\overline{T},1} = 75 \quad n_{F}^{\overline{T},2} = 38$

$\overline{D}_{F}^{\overline{T}} = 1.84041\,\%$ (1.88273 %) $\quad S_{F}^{\overline{T}} = 1.63466\,\%$ (0.94985 %) $\quad C_{F}^{\overline{T}} = 0.97044$ (0.94095)

Predicted price
Observed price

Note : The number in the parenthesis under the value is obtained by recalculating the value based on the 10 trading days before $t_{n_{F}^{\overline{T},1}+1}$.

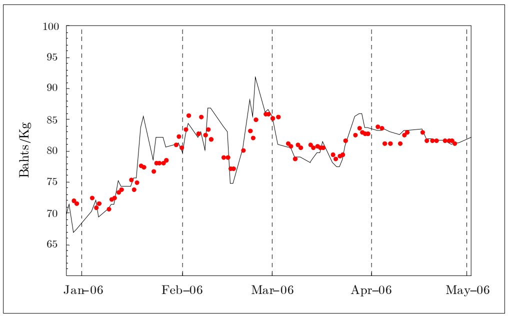
RSRS3-JUN06



$\mathbf{F} = \mathbf{RSRS3}$ $\quad \overline{T} = \mathrm{JUN06}$ $\quad n_{\mathrm{F}}^{\overline{T}} = 109$ $\quad n_{\mathrm{F}}^{\overline{T},1} = 75$ $\quad n_{\mathrm{F}}^{\overline{T},2} = 34$

$\overline{D}_{\mathrm{F}}^{\overline{T}} = 1.92899\,\%$ (1.21130 %) $\quad S_{\mathrm{F}}^{\overline{T}} = 1.71780\,\%$ (0.94608 %) $\quad C_{\mathrm{F}}^{\overline{T}} = 0.94884$ (0.93994)

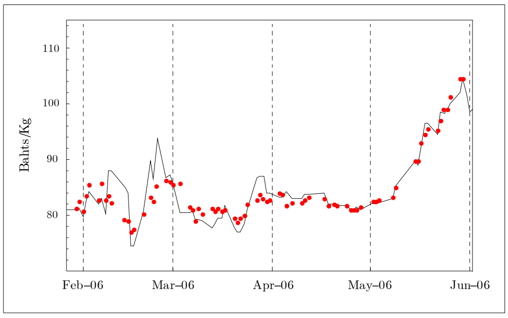
RSRS3-JUL06



$\mathbf{F} = \mathbf{RSRS3}$ $\quad \overline{T} = \mathrm{JUL06}$ $\quad n_{\mathrm{F}}^{\overline{T}} = 113$ $\quad n_{\mathrm{F}}^{\overline{T},1} = 76$ $\quad n_{\mathrm{F}}^{\overline{T},2} = 37$

$\overline{D}_{\mathrm{F}}^{\overline{T}} = 2.34340\,\%$ (0.55550 %) $\quad S_{\mathrm{F}}^{\overline{T}} = 2.08780\,\%$ (0.45537 %) $\quad C_{\mathrm{F}}^{\overline{T}} = 0.86134$ (0.77732)

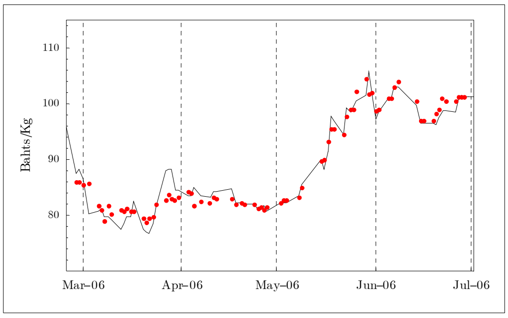
RSRS3-AUG06



$\mathbf{F} = \mathbf{RSRS3}$ $\quad \overline{T} = \mathrm{AUG06}$ $\quad n_{\mathrm{F}}^{\overline{T}} = 111$ $\quad n_{\mathrm{F}}^{\overline{T},1} = 74$ $\quad n_{\mathrm{F}}^{\overline{T},2} = 37$

$\overline{D}_{\mathrm{F}}^{\overline{T}} = 1.95700\,\%$ (1.12520 %) $\quad S_{\mathrm{F}}^{\overline{T}} = 1.99640\,\%$ (0.66193 %) $\quad C_{\mathrm{F}}^{\overline{T}} = 0.91919$ (0.95109)

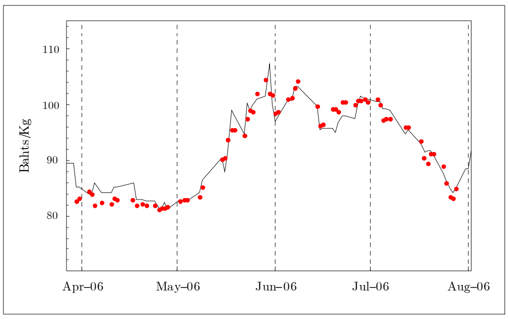
RSRS3-SEP06



$\mathbf{F} = \mathbf{RSRS3}$ $\quad \overline{T} = \mathrm{SEP06}$ $\quad n_{\mathrm{F}}^{\overline{T}} = 113$ $\quad n_{\mathrm{F}}^{\overline{T},1} = 75$ $\quad n_{\mathrm{F}}^{\overline{T},2} = 38$

$\overline{D}_{\mathrm{F}}^{\overline{T}} = 1.54877\,\%$ (1.26095 %) $\quad S_{\mathrm{F}}^{\overline{T}} = 1.43193\,\%$ (0.87299 %) $\quad C_{\mathrm{F}}^{\overline{T}} = 0.97695$ (0.87215)

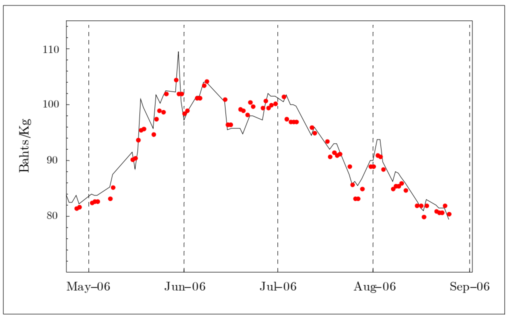
RSRS3-OCT06



$\mathbf{F} = \mathbf{RSRS3}$ $\quad \overline{T} = \mathrm{OCT06}$ $\quad n_{\mathrm{F}}^{\overline{T}} = 91$ $\quad n_{\mathrm{F}}^{\overline{T},1} = 71$ $\quad n_{\mathrm{F}}^{\overline{T},2} = 20$

$\overline{D}_{\mathrm{F}}^{\overline{T}} = 1.45259\,\%$ (1.03748 %) $\quad S_{\mathrm{F}}^{\overline{T}} = 1.22290\,\%$ (0.65212 %) $\quad C_{\mathrm{F}}^{\overline{T}} = 0.97543$ (0.96985)

RSRS3-NOV06



$\mathbf{F} = \mathbf{RSRS3}$ $\quad \overline{T} = \mathrm{NOV06}$ $\quad n_{\mathrm{F}}^{\overline{T}} = 75$ $\quad n_{\mathrm{F}}^{\overline{T},1} = 75$ $\quad n_{\mathrm{F}}^{\overline{T},2} = 0$

$\overline{D}_{\mathrm{F}}^{\overline{T}} = 1.81982\,\%$ (0.97570 %) $\quad S_{\mathrm{F}}^{\overline{T}} = 1.28111\,\%$ (0.31130 %) $\quad C_{\mathrm{F}}^{\overline{T}} = 0.96425$ (0.88810)

Predicted price
Observed price

Note : The number in the parenthesis under the value is obtained by recalculating the value based on the 10 trading days before $t_{n_{\mathrm{F}}^{\overline{T},1}+1}$.

3.6.4 Discussion

This subsection discusses about the price differences and the correlations between the observed futures prices and the predicted futures prices in WR5 case and RSRS3 case by referring to the results obtained in the previous subsection. We start by considering the price differences. In WR5 case, supposing that the prices of WR5 were stable at 10 Bahts/kg, the maximum value of $\overline{D}_{\mathbf{F}}^{\overline{T}}$ s is about 3% or about 0.3 Baht/kg for the WR5-contracts. In RSRS3 case, supposing that the prices of RSRS3 were stable at 80 Bahts/kg, the maximum value of $\overline{D}_{\mathbf{F}}^{\overline{T}}$ s is about 2.35% or about 1.88 Baths/kg for the RSRS3-contracts. Nevertheless, for all selected futures contracts, $S_{\mathbf{F}}^{\overline{T}}$ s are relatively high and this implies that $\overline{D}_{\mathbf{F}}^{\overline{T}}$ s are not significantly different from zero. Namely, the observed futures prices and the predicted futures prices obtained from our model are not significantly different.

It can be noticed from the graphs that, for each selected futures contract, the price differences are hardly realized on the days t_n s close to $t_{n_{\mathbf{F}}^{\overline{T},1}+1}$, the first element of $P_{\mathbf{F}}^{\overline{T},2}$. In other words, for each selected futures contract, if we consider its futures prices only on the 10 trading days before $t_{n_{\mathbf{F}}^{\overline{T},1}+1}$ and recalculate $\overline{D}_{\mathbf{F}}^{\overline{T}}$ then we have seen that the new value is smaller than the previous value (except on the futures contracts, WR5-MAR05, RSRS3-OCT05, and RSRS3-MAY06). These results indicate that if the predicted futures prices were very close to the exact no-arbitrage futures prices[6], it is difficult for the market participants who traded a futures contract of WR5 or RSRS3 in the sample periods to take the arbitrage opportunities from the futures market, especially, on the trading days close to the maturity date of the futures contract. This remark can be clearly seen, for examples, in the futures contracts: WR5-MAY05, WR5-JUL05, WR5-SEP05, RSRS3-MAR05, RSRS3-JUL05, RSRS3-DEC05, RSRS3-MAR06, and RSRS3-JUL06.

Finally, we consider the relationships between the observed futures prices and the predicted futures prices for WR5 case and RSRS3 case. It can be seen in each selected futures contract that the two sequences are strongly positive correlated. In the WR5 case, $C_{\mathbf{F}}^{\overline{T}}$ s are higher than 0.54 and, in the case of RSRS3, $C_{\mathbf{F}}^{\overline{T}}$ s are higher than 0.63. In fact, $C_{\mathbf{F}}^{\overline{T}}$ s are higher than 0.9 almost all selected futures contracts for WR5 case and RSRS3 case. Namely, the two sequences are almost perfectly correlated. In the economics point of view, these just obtained results combined with the results obtained in the previous paragraph can be explained by the equilibrium in the futures markets, namely, for each futures market, the observed futures prices approach their corresponding no-arbitrage futures prices when the futures market is close to the equilibrium.

[6] Suppose that the true-parameters are known. Thus, the exact no-arbitrage futures prices can be computed.

3.7 Backwardation and Contango in AFET

We first recall the relation between the commodity spot prices and their corresponding no-arbitrage futures prices as expressed in Proposition 5, i.e.,

$$F^T(t, S_t, \delta_t; \theta) = S_t e^{A(T-t;\theta)+B(T-t;\theta)\delta_t}. \tag{3.7.1}$$

From relation (3.7.1), one can see that the futures price could be greater or less than the commodity spot price, depending on the sign of the term $A(T-t;\theta) + B(T-t;\theta)\delta_t$.

Definition 3.1 (Backwardation and Contango)

For given t, S_t, and δ_t, if $S_t > F^T(t)$ at a maturity date $T > t$, we say that the futures market exhibits <u>*backwardation*</u> at the maturity date T. Conversely, if $S_t < F^T(t)$ at a maturity date $T > t$, we say that the futures market exhibits <u>*contango*</u> at the maturity date T.

It should be noted that the futures market is in backwardation if the convenience yield is sufficiently high, while it is in contango if the convenience yield is sufficiently low. One can verify that if the futures price is strictly deceasing in T, $\partial F^T / \partial T < 0$, the futures market will be in backwardation. On the other hand, if the futures price is strictly increasing in T, $\partial F^T / \partial T > 0$, the futures market will be in contango. Moreover, for given t, S_t, and δ_t, we can observe backwardation and contango situations in a futures market by considering forward curves or forward surfaces for the commodity defined as follows.

Definition 3.2 (Commodity Forward Curve and Forward Surface)

The <u>*forward curve*</u> prevailing on day t for a given commodity is a two-dimensional graphical representation of the set $\{(T, F^T(t, S_t, \delta_t; \theta)); T > t\}$ of futures prices for different traded maturity dates T. By analogy to the definition of the forward curve, the <u>*forward surface*</u> prevailing on a time period $[0, \overline{T}]$, $\overline{T} > 0$, for the commodity is a three-dimensional surface representation of the set $\{(t, T_t, F^{T_t}(t, S_t, \delta_t; \theta)); t \le T_t < \infty, t \in [0, \overline{T}]\}$ of futures prices for different traded maturity dates T_t.

The forward curve observed on a trading day t and the forward surface observed on the time period $[0, t]$ are important tools to see how the market prices the commodity for various delivery dates. Namely, the forward curve and the forward surface tell the market participants where the market sees the commodity spot prices in the future. Moreover, it provides the marking-to-the market to date of a portfolio of forward contracts which is very useful for capital budgeting decisions of investors in the futures market.

Using relation (3.7.1) and (D05)-(D06) in Appendix D, we have

$$\frac{1}{F^T}\frac{\partial F^T}{\partial T} = (p_1 B^2(T-t) + p_2 B(T-t) + p_3 - 1)\delta_t$$
$$+q_1 B^2(T-t) + (\alpha_T(t) + q_2)B(T-t) + q_3, \qquad (3.7.2)$$

where p_i, q_i, $i = 1,\ldots,3$, are constants given in Appendix D. Equation (3.7.2) indicates that the situations known as backwardation and contango in a futures market at a maturity date T depend on the sign of the return of the futures contract at time t. This futures return depends on the convenience yield δ_t, the seasonal function $\alpha_T(t)$, and the current time t.

Finally, we close this chapter by illustrating forward surfaces for WR5 and RSRS3 in the sample periods. Using the estimated parameters of WR5 tabulated in Table 3.4 and the futures prices data of WR5, $F^{T_w}_{N_w}$, we compute the forward surface for WR5 in the sample period 26/08/2004 - 26/08/2005 and display it in Figure 3.9. Likewise, the forward surface of RSRS3 in the sample period 26/08/2004 - 26/08/2006 is computed using the estimated parameters of RSRS3 tabulated in Table 3.4 and the futures prices data of RSRS3, $F^{T_R}_{N_R}$. The forward surface for RSRS3 is displayed in Figure 3.10.

It can be clearly seen from the forward surface for WR5 (Figure 3.9) that the futures market of WR5 exhibited backwardation almost every trading day for those contracts having maturity dates in August 2005 (approximately one year from the first observation day). This means the market expected that there would be high positive yields for WR5 in the period August 2005 to September 2005. These yields can be observed from the extracted convenience yields of WR5 shown in Figures 3.6 and they were apparently high in that time. Look at the futures contracts having maturity dates in March 2005. WR5 futures market was in contango since the market expected that there would be a high inventory cost in that time. As seen in Figure 3.6, yields were negative in March 2005 and this is in accordance with the market expectation.

In contrast to WR5 futures market, it can be clearly seen from the forward surface for RSRS3 (Figure 3.10) that RSRS3 futures market exhibited contango almost every trading day for those contracts having maturity dates in August 2005 and August 2006 (approximately one year and two years from the first observation day). This means the market saw that yields would decline in the period August to September of the two observation years. From the extracted convenience yields of RSRS3 shown in Figure 3.8, one can notice that the yields were decreasing in these periods and this follows the market expectation.

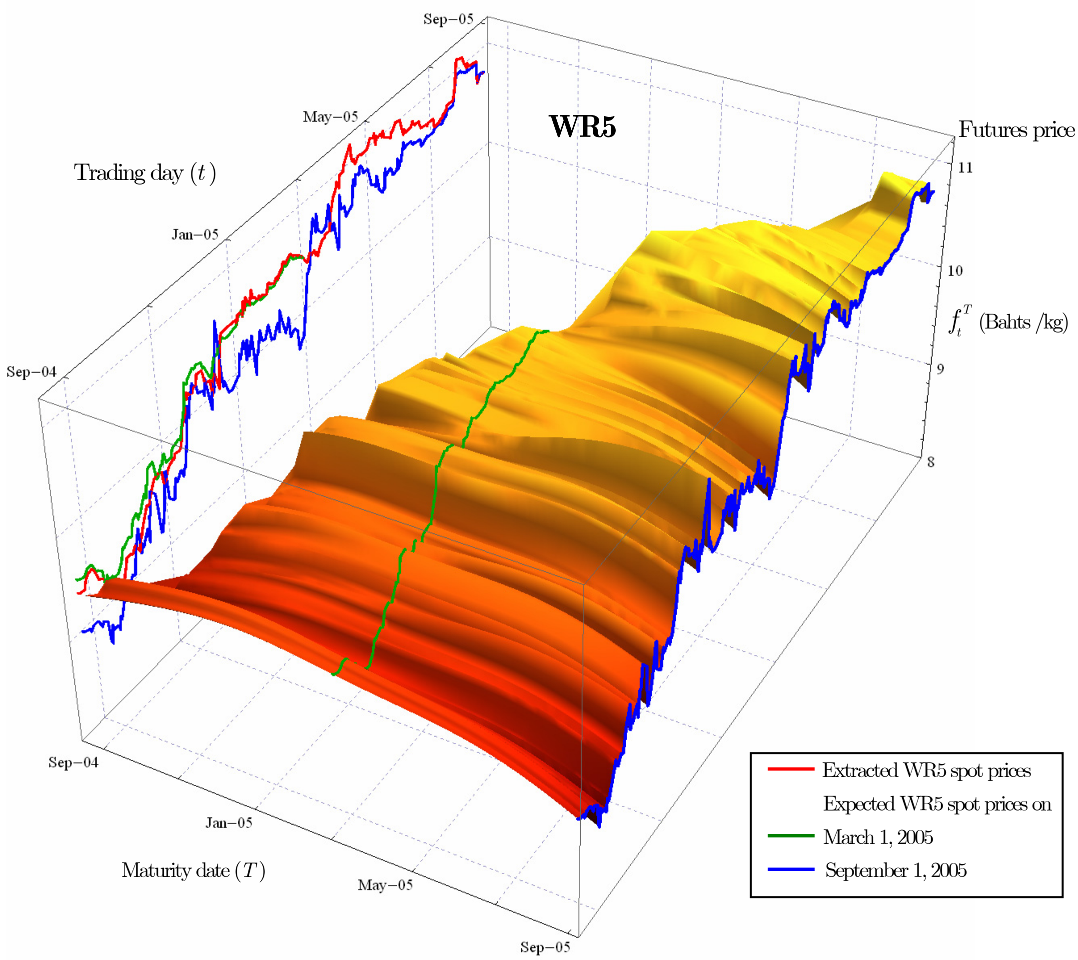


Figure 3.9: The forward surface for WR5 in the sample period 26/08/2004 - 26/08/2005 obtained from our model. The points on the surface are of the form (t, T, f_t^T) where f_t^T is the expectation of the WR5 spot price on day T computed by using the information on day t, i.e.,

$$f_t^T = E_{\mathbb{Q}}[S_T \mid \mathcal{F}_t] = E_{\mathbb{Q}}[S_T \mid S_t, \delta_t].$$

In fact, f_t^T is the no-arbitrage WR5 futures price on day t of the futures contract having maturity date T. The blue curve on the right boundary of the forward surface represents the evolution of the expectations of the WR5 spot price on September 1, 2005 in the sample period and this blue curve is projected onto the left plane. The green curve lies on the forward surface represents the evolution of the expectations of the WR5 spot price on March 1, 2005 in the sample period and this green curve is projected onto the left plane. The red curve lying on the left plane represents the extracted WR5 spot prices in the sample period. WR5 futures market exhibited backwardation almost every trading day for those contracts having maturity dates in August 2005. On the other hand, WR5 futures market was in contango almost every trading day for those contracts having maturity dates in March 2005.

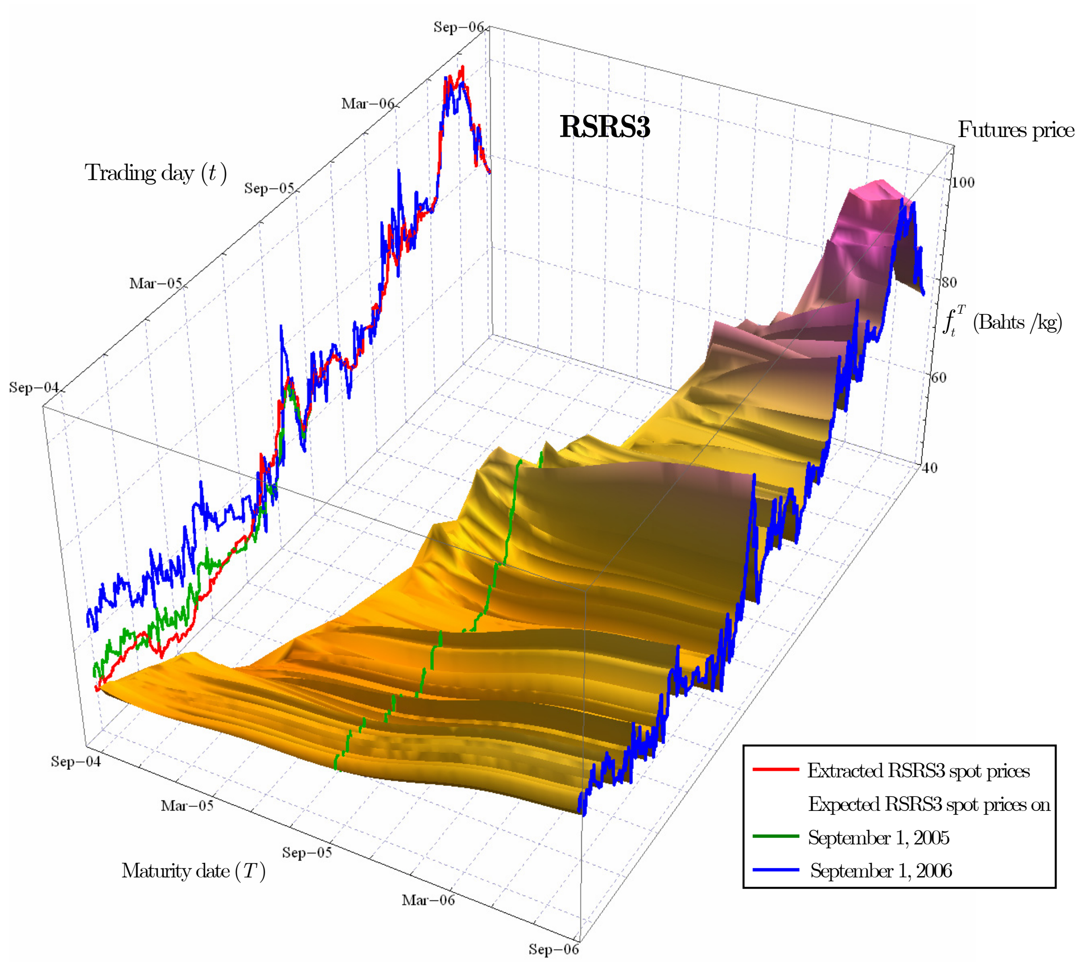


Figure 3.10: The forward surface for RSRS3 in the sample period 26/08/2004 - 26/08/2006 obtained from our model. The points on the surface are of the form (t, T, f_t^T) where f_t^T is the expectation of the RSRS3 spot price on day T computed by using the information on day t, i.e.,

$$f_t^T = E_{\mathbb{Q}}[S_T \mid \mathcal{F}_t] = E_{\mathbb{Q}}[S_T \mid S_t, \delta_t].$$

In fact, f_t^T is the no-arbitrage RSRS3 futures price on day t of the futures contract having maturity date T. The blue curve on the right boundary of the forward surface represents the evolution of the expectations of the RSRS3 spot price on September 1, 2006 in the sample period and this blue curve is projected onto the left plane. The green curve lies on the forward surface represents the evolution of the expectations of the RSRS3 spot price on September 1, 2005 in the sample period and this green curve is projected onto the left plane. The red curve lying on the left plane represents the extracted RSRS3 spot prices in the sample period. RSRS3 futures market exhibited contango almost every trading day for those contracts having maturity dates in August 2005 and August 2006.

Conclusions and Outlook

In this dissertation, we have introduced a two-factor stochastic model of commodity prices which is an extension of the model proposed by Nielsen-Schwartz (2004) [N-02]. The first factor is the commodity spot price which follows a GBM with a time-varying volatility. The second factor is the instantaneous convenience yield which follows an extended CIR process by adding a time-dependent function into the drift term of the process in order to describe seasonal variations in commodity prices. The time-varying volatilities of the commodity spot prices and the instantaneous convenience yields are proportional to the square root of the instantaneous convenience yields. Our modeling concerns about two important things: a link between price volatilities and convenience yields as suggested by the theory of storage, and seasonality in commodity prices and convenience yield volatilities.

In terms of volatilities of the commodity prices and the convenience yields, we imposed sufficient conditions for the inaccessibility to nonpositive values of the volatility process. In terms of pricing futures and futures options, by supposing that the market is arbitrage-free, we derived closed-form solutions for the futures prices. The closed-form solutions are consistent with the theory of storage: futures prices tend to be lower than spot prices when convenience yields are sufficiently high and vice versa. In addition, the closed-form solutions lead to the extraction formulas for the two factors, commodity spot prices and instantaneous convenience yields, under the assumption that two no-arbitrage futures prices having different maturities can be observed. Moreover, numerical solutions for European futures options prices were derived using a method of Fourier transforms.

In terms of estimating model parameters, we used the maximum likelihood approach to get estimate model parameters. We derived a closed-form approximation to the log-likelihood function of the log-futures prices data. Applying the closed-form approximation to the daily futures prices data of the two agricultural commodities in Thailand, rice and natural rubber, provided on the website of AFET, we then obtained the corresponding approximate MLEs. Plugging the estimated parameters into the extraction formulas gave us the time series of the spot prices and the instantaneous convenience yields of the two commodities in the sample periods. The time series show a clear seasonal pattern in both prices and convenience yield volatilities of the two commodities. The numerical results suggest that convenience yields tend to be high when inventory/supply is low, and vice versa. These results support the conceptual ideas in the theory of storage. However, there is an impact from the Thai price intervention scheme on the domestic rice prices which can be observed from the daily extracted rice price volatilities such that they are less variable than the daily extracted rubber price volatilities.

In terms of the practical applicability of our model, we computed the price differences and the correlations between the observed futures prices and their corresponding no-arbitrage (predicted) futures prices, obtained by plugging the estimated parameters into the extraction formulas, for several futures contracts of the two commodities. The results obtained show the price differences are insignificantly different from zero and the correlations are highly positive. This implies that our model is applicable for the two commodities prices. Furthermore, we have observed that, for each selected futures contract, the price differences on the days close to its maturity date are hardly realized by the market participants. In the economic point of view, these results can be explained by the equilibrium in the futures market, namely, the observed futures prices approach their corresponding no-arbitrage futures prices when the futures market is close to the equilibrium. Finally, we analyzed the implications of our model for capital budgeting decisions by investigating the situations known as backwardation and contango in AFET. The forward surfaces for the two commodities in the sample periods are displayed and we have found that, for long maturity futures contracts, the futures market of rice exhibited backwardation, while the futures market of natural rubber exhibited contango. These results indicate that, in the long run futures of the two commodities prices, the market has expected a decrease in rice prices, but an increase in natural rubber prices.

Our model can be extended to model agricultural and other seasonal commodities in the case that the commodity spot prices exhibit sudden and unexpected price jumps. Under this consideration, one can modify our model to a stochastic volatility/jump-diffusion model as proposed by Bates (1996) [B-02]:

$$\begin{aligned} dS_t &= (r - \delta_t + \lambda_S(\beta_1\delta_t + \beta_2) - \lambda_J\bar{k})\,S_t dt + \sqrt{\beta_1\delta_t + \beta_2}\;S_t\,dW_t^{(1)} + K\,dJ_t \\ d\delta_t &= (\alpha_T(t) - \kappa\delta_t + \lambda_\delta(\beta_1\delta_t + \beta_2))dt + \sigma_\delta\sqrt{\beta_1\delta_t + \beta_2}\;dW_t^{(2)}, \end{aligned} \qquad \text{(M*)}$$

where

$$\text{Prob}(dJ_t = 1) = \lambda_J\,dt, \quad \ln(1+K) \sim \textbf{normal}\left(\ln(1+\bar{k}) - \tfrac{1}{2}\delta_J^2, \delta_J^2\right),\; E[K] = \bar{k},\; \delta_J > 0,$$

K is the random percentage jump conditioned upon a Poisson distribution J_t occurring with intensity λ_J, and $\ln(1+K)$ is normally distributed random variable. Price jumps will typically occur due to abrupt changes in supply-demand conditions or interventions of government price support scheme. Such discontinuities in the price path of a commodity will affect futures and/or futures option prices. Using model (M*) and following the approach in Bates (1996) [B-02], one can derive futures and/or futures option prices for commodities under a unique equivalent martingale measure in a jump-diffusion setting by considering a specific equilibrium model. This will be a future work of a great interest for researchers in the field of commodity price modeling.

Appendices

Appendix A

Derivation of Model (1.2.1)

The model (1.2.1) is derived based on the stochastic volatility problem proposed in Hull-White (1987) [H-04]. Consider a derivative asset f in which its price at a current time t depends upon the commodity spot price S_t and the instantaneous convenience yield δ_t, i.e., $f \equiv f(t, S_t, \delta_t), t \in [0, T]$. Under an original probability measure $\mathbb{P}$, we assume that the dynamics of the commodity spot prices and the instantaneous convenience yields satisfy the following stochastic differential equations (SDEs):

$$\left.\begin{aligned} dS_t &= \mu_S(t, S_t, \delta_t)dt + \sigma_S(t, S_t, \delta_t)d\hat{W}_t^{(1)}, \\ d\delta_t &= \mu_\delta(t, S_t, \delta_t)dt + \sigma_\delta(t, S_t, \delta_t)d\hat{W}_t^{(2)}, \end{aligned}\right\} \tag{A01}$$

where $\hat{W} = (\hat{W}_t^{(1)}, \hat{W}_t^{(2)})$ is a two-dimensional Brownian motion under $\mathbb{P}$, ρ is a constant correlation between the two Brownian motions, μ_S, μ_δ, σ_S, and σ_δ are some suitable functions. Suppose S_t and δ_t are non-traded assets and there are no assets that are clearly instantaneously perfectly correlated with the state variables S_t and δ_t. Thus, it does not seem possible to form a hedge portfolio that eliminates all the risk. However, as was shown by Garman (1976) [G-01], under the general partial equilibrium conditions which imply the absence of riskless arbitrage opportunities in continuous markets, f must satisfy the following fundamental differential equation (FDE):

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma_S^2\frac{\partial^2 f}{\partial S^2} + \rho\sigma_S\sigma_\delta\frac{\partial^2 f}{\partial S\partial\delta} + \frac{1}{2}\sigma_\delta^2\frac{\partial^2 f}{\partial\delta^2} + (\mu_S + \Lambda_S)\frac{\partial f}{\partial S} + (\mu_\delta + \Lambda_\delta)\frac{\partial f}{\partial\delta} - rf = 0, \tag{A02}$$

where r is the risk free interest rate, and the two functions Λ_S and Λ_δ are known as the market risk-aversions (possibly random variables) depending on σ_S, σ_δ, and a market kernel $\mathrm{K}(t, S_t, \delta_t; T, S_T, \delta_T)$ which is not necessarily unique (see Garman (1976) [G-01] for the derivations of the market kernel and the FDE (A02)).

Next, we define a process $W = (W_t^{(1)}, W_t^{(2)})$ by

$$W_t^{(1)} := \hat{W}_t^{(1)} - \int_0^t \left(\Lambda_S\,\sigma_S^{-1}\right)_s ds, \tag{A03}$$

$$W_t^{(2)} := \hat{W}_t^{(2)} - \int_0^t \left(\Lambda_\delta\,\sigma_\delta^{-1}\right)_s ds, \tag{A04}$$

for $t \in [0, T]$, where the integral terms contained in (A03)-(A04) are assumed finite $\mathbb{P}$- a.s. . Under the regular conditions, applying the Girsanov theorem (Theorem 3.5.1 in Karatzas-Shreve (1988) [K-01]) to the process $\hat{W}$ gives us the process W is a two-dimensional Brownian motion under a probability measure $\mathbb{Q} \sim \mathbb{P}$.

Using (A03)-(A04), the SDEs (A01) can be written as follows:

$$\left.\begin{aligned} dS_t &= (\mu_S(t,S_t,\delta_t) + \Lambda_S)dt + \sigma_S(t,S_t,\delta_t)dW_t^{(1)}, \\ d\delta_t &= (\mu_\delta(t,S_t,\delta_t) + \Lambda_\delta)dt + \sigma_\delta(t,S_t,\delta_t)dW_t^{(2)}. \end{aligned}\right\} \tag{A05}$$

Applying Theorem 1 in Heath-Schweizer (2000) [H-01] to the SDEs (A05) and using the FDE (A02), we get that the no-arbitrage price of the derivative asset satisfies

$$f(t,S_t,\delta_t) = E_{\mathbb{Q}}[e^{-r(T-t)} f(T,S_T,\delta_T) \mid S_t,\delta_t], \tag{A06}$$

for all $(t,S_t,\delta_t) \in [0,T] \times D,$ where D is the domain of the diffusion process $(S_t,\delta_t)_{t\in[0,T]}$.

The just obtained result (A06) tells us that the no-arbitrage price of a derivative asset at a current time t is the present value of the expected payoff of the derivative at the maturity date T of the derivative conditioned on the information at the current time t. Next, we define a random process $X = (X_t)$ by

$$X_t := e^{r(T-t)} f(t,S_t,\delta_t) \tag{A07}$$

for $t \in [0,T]$. We see by the definition (A07) that X_t is in fact the future payoff at the maturity date T of the derivative at a current time t. Applying the Itô formula to (A07) under $\mathbb{Q}$ gives us the dynamics of X as follows:

$$\begin{aligned} dX_t &= e^{r(T-t)}\left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma_S^2\frac{\partial^2 f}{\partial S^2} + \rho\sigma_S\sigma_\delta\frac{\partial^2 f}{\partial S\partial\delta} + \frac{1}{2}\sigma_\delta^2\frac{\partial^2 f}{\partial\delta^2} + (\mu_S + \Lambda_S)\frac{\partial f}{\partial S} + (\mu_\delta + \Lambda_\delta)\frac{\partial f}{\partial\delta} - rf\right)dt \\ &\quad + e^{r(T-t)}\frac{\partial f}{\partial S}\sigma_S\,dW_t^{(1)} + e^{r(T-t)}\frac{\partial f}{\partial\delta}\sigma_\delta\,dW_t^{(2)}. \end{aligned} \tag{A08}$$

Using the FDE (A02), one can easily see that the drift term of X in (A08) is equal to zero and this implies that X is a martingale under $\mathbb{Q}$. This in turn leads to the conclusion that, under the no-arbitrage assumptions, there exists a probability measure $\mathbb{Q}$ such that the process of the future payoff at the maturity date T of the derivative is a martingale under $\mathbb{Q}$. Therefore, $\mathbb{Q}$ is called an equivalent martingale measure or a risk-neutral probability measure.

Now, we return to the SDEs (A05). In the literature of commodity price modeling, there are several choices to model the drifts and the diffusion coefficients of the processes (see Lautier (2003) [2003] for the review of commodity price modeling). In this research, we extent the model proposed by Nielsen-Schwartz (2004) [N-02]. The seasonal function $\alpha_T(t)$ is added into the drift term of the convenience yield process in order to describe seasonal variations in commodity prices and convenience yield volatilities. As previously mentioned, the market kernel may not unique and this leads to the problem of incomplete market. However, in order to uniquely determine the price of the derivative, we must impose some conditions on the market risk-aversions Λ_S and Λ_δ. As suggested by Cox-Ingersoll-Ross (1985) [C-02], we assume that

$$\Lambda_S \propto (\beta_1\delta_t + \beta_2)S_t \Rightarrow \Lambda_S = \lambda_S(\beta_1\delta_t + \beta_2)S_t, \tag{A09}$$

and

$$\Lambda_\delta \propto (\beta_1\delta_t + \beta_2) \Rightarrow \Lambda_\delta = \lambda_\delta(\beta_1\delta_t + \beta_2), \tag{A10}$$

for some real constants λ_S and λ_δ.

Appendix B

Proof of Proposition 1

We need the following result which is Example 1 in the proof of a Comparison Theorem for Solution of Stochastic Differential Equations proposed by Zhiyuan (1984) [Z-01]:

Example 1. Consider two SDEs with the same diffusion coefficient σ :

$$\begin{cases} dX_t^{(i)} = b^{(i)}(t, W_t, X_t^{(i)})dt + \sigma(t, W_t, X_t^{(i)})dW_t, \\ X_0^{(i)} = x_0^{(i)}, \quad i, = 1, 2. \end{cases} \tag{B01}$$

where $\sigma, b^{(1)}, b^{(2)} : G \subseteq [0, \infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and locally Lipschitz continuous with respect to x and $x_0^{(1)} \le x_0^{(2)}$, $b^{(1)} \le b^{(2)}$ in G. Then

$$X_t^{(1)} \le X_t^{(2)}, \quad \text{a.s. for all } t \ge 0. \tag{B02}$$

There are two parts of the proof of Proposition 1:

<u>Part 1</u> (Sufficient conditions for the inaccessibility of $\hat{\delta}_t$ to nonpositive values)

Consider the dynamics of $\hat{\delta}_t$, i.e.,

$$d\hat{\delta}_t = (\upsilon_T(t) - (\kappa - \lambda_\delta \beta_1)\hat{\delta}_t)dt + \sigma_\delta \beta_1 \sqrt{\hat{\delta}_t}\, dW_t^{(2)}, \tag{B03}$$

where

$$\upsilon_T(t) = \beta_1 \alpha_T(t) + \kappa \beta_2. \tag{B04}$$

We need sufficient conditions to ensure that $\hat{\delta}_t > 0$ $\mathbb{Q}$-a.s. for all $t \in [0, T]$.

Firstly, we construct a CIR process of the following form:

$$\begin{cases} dy_t = (\gamma_y(\theta) - (\kappa - \lambda_\delta \beta_1) y_t)\, dt + \sigma_\delta \beta_1 \sqrt{y_t}\, dW_t^{(2)}, \\ y_0 = \hat{\delta}_0, \end{cases} \tag{B05}$$

where $\gamma_y(\theta)$ is a constant depending only on the parameters and satisfies the condition

$$0 < \gamma_y(\theta) \le \inf_{t \in [0,T]} \upsilon_T(t; \theta). \tag{B06}$$

Since we have

$$y_0 = \hat{\delta}_0 > 0, \tag{B07}$$

the condition

$$\frac{\gamma_y(\theta)}{\sigma_\delta^2 \beta_1^2} \ge \frac{1}{2} \tag{B08}$$

guarantees that $$y_t > 0 \ \ \mathbb{Q}\text{-a.s. for all } t \in [0,T], \tag{B09}$$

(see Ikeda-Watanabe (1981) [I-01] on page 221). Note that the conditions (B07) and (B08) are two sufficient conditions to ensure that $y_t > 0$ $\mathbb{Q}$-a.s. for all $t \in [0,T]$.
It is not difficult to see that

$$b^{(1)}(t,w,y) := \gamma_y(\theta) - (\kappa - \lambda_\delta \beta_1)y \leq \upsilon_T(t;\theta) - (\kappa - \lambda_\delta \beta_1)y =: b^{(2)}(t,w,y),$$

for all $(t,\omega,y) \in G = [0,\infty) \times \mathbb{R} \times \mathbb{R}^+$. Since $\sigma(t,\omega,y) := \sigma_\delta \beta_1 \sqrt{y}$ is locally Lipschitz continuous in y on $\mathbb{R}^+$. Hence, from Example 1 and (B03)-(B05), we have

$$y_t \leq \hat{\delta}_t \ \ \mathbb{Q}\text{-a.s. for all } t \in [0,T]. \tag{B10}$$

Combining (B09) and (B10) leads to the conclusion that

$$\hat{\delta}_t > 0 \ \ \mathbb{Q}\text{-a.s. for all } t \in [0,T]. \tag{B11}$$

In order to choose $\gamma_y(\theta)$ to satisfy (B06), we first consider

$$\begin{aligned}\inf_{t\in[0,T]} \alpha_T(t;\theta) &= \alpha_0 + \inf_{t\in[0,T]} \left(f_\alpha(T-t;\theta) \sum_{k=1}^{K^\alpha} \alpha_k^{(1)} \cos(2\pi k t) + \alpha_2^{(k)} \sin(2\pi k t) \right) \\ &\geq \alpha_0 + \sum_{k=1}^{K^\alpha} \inf_{t\in[0,T]} \left(\alpha_k^{(1)} f_\alpha(T-t;\theta)\cos(2\pi k t)\right) + \sum_{k=1}^{K^\alpha} \inf_{t\in[0,T]} \left(\alpha_k^{(2)} f_\alpha(T-t;\theta)\sin(2\pi k t)\right).\end{aligned} \tag{B12}$$

At the end of Appendix D, we show that $f_\alpha(T-t;\theta)$ is a positive strictly increasing function in t. This implies

$$\inf_{t\in[0,T]} \left(\alpha_k^{(1)} f_\alpha(T-t;\theta)\cos(2\pi k t)\right) \geq f_\alpha(T;\theta) \inf_{t\in[0,T]} \left(\alpha_k^{(1)} \cos(2\pi k t)\right) = -f_\alpha(T;\theta)\left|\alpha_k^{(1)}\right|, \tag{B13}$$

$$\inf_{t\in[0,T]} \left(\alpha_k^{(2)} f_\alpha(T-t;\theta)\sin(2\pi k t)\right) \geq f_\alpha(T;\theta) \inf_{t\in[0,T]} \left(\alpha_k^{(2)} \sin(2\pi k t)\right) = -f_\alpha(T;\theta)\left|\alpha_k^{(2)}\right|, \tag{B14}$$

for all $k = 1,\ldots,K^\alpha$. From (B04), applying the inequalities (B13) and (B14) to (B12) yields

$$\inf_{t\in[0,T]} \upsilon_T(t;\theta) = \beta_1 \inf_{t\in[0,T]} \alpha_T(t;\theta) + \kappa\beta_2 \ \geq \beta_1 \left(\alpha_0 - f_\alpha(T;\theta) \sum_{k=1}^{K^\alpha} \left|\alpha_k^{(1)}\right| + \left|\alpha_k^{(2)}\right| \right) + \kappa\beta_2. \tag{B15}$$

We choose

$$\gamma_y(\theta) = \beta_1 \left(\alpha_0 - f_\alpha(T;\theta) \sum_{k=1}^{K^\alpha} \left|\alpha_k^{(1)}\right| + \left|\alpha_k^{(2)}\right| \right) + \kappa\beta_2. \tag{B16}$$

Substituting $\gamma_y(\theta)$ in (B08) with the RHS of (B16) gives us the sufficient condition for the inaccessibility of $\hat{\delta}_t$ to nonpositive values as written in the expression (1.2.10).

Part 2 (The uniqueness of the strong solutions of the SEDs (1.2.1) which do not explode)

The dynamics of S_t and δ_t can be rewritten as

$$dS_t = (r - \delta_t + \lambda_S(\beta_1\delta_t + \beta_2))\,S_t dt + \sqrt{\beta_1\delta_t + \beta_2}\;S_t\,dW_t^{(1)}, \tag{B17}$$

$$d\delta_t = b(t,S_t,\delta_t)dt + a(t,S_t,\delta_t)\,dW_t^{(2)}, \tag{B18}$$

where

$$b(t,S_t,\delta_t) := (\alpha_T(t) + \lambda_\delta\beta_2) + (\lambda_\delta\beta_1 - \kappa)\delta_t, \tag{B19}$$

$$a(t,S_t,\delta_t) := \sigma_\delta\sqrt{\beta_1\delta_t + \beta_2}, \tag{B20}$$

$$\alpha_T(t) = \alpha_0 + f_\alpha(T-t;\theta)\left(\sum_{k=1}^{K^\alpha}\left(\alpha_k^{(1)}\cos(2\pi kt) + \alpha_k^{(2)}\sin(2\pi kt)\right)\right), \tag{B21}$$

$$f_\alpha(T-t;\theta) = \frac{\sqrt{p} + p_2 + (\sqrt{p} - p_2)\,e^{\sqrt{p}(T-t)}}{e^{\sqrt{p}(T-t)}}. \tag{B22}$$

It is easy to see that the drifts and the diffusion coefficients of S_t and δ_t are C^1 in (t,S,δ) on $[0,T]\times D$, where $D := (0,\infty)\times(\frac{-\beta_2}{\beta_1},\infty)$. These results imply

"the strong uniqueness holds for the SDEs (B17) and(B18)". (B23)

Namely, there exists a unique strong solution $X \equiv (S_t,\delta_t)_{t\in[0,T]}$ of the SDEs (B17) and (B18) with $X_0 = (S_0,\delta_0) \in D$ (see Theorem 5.2.5 in Karatzas-Shreve (1988) [K-01]).

To ensure that δ_t does not explode $\mathbb{Q}$-a.s. for all $t \in [0,T]$, we must show that the following linear growth conditions hold for the drift and the diffusion coefficient of δ_t, i.e.,

$$\left|b(t,x)\right|^2 \le K^2(1+|x|^2), \tag{B24}$$

$$\left|a(t,x)\right|^2 \le K^2(1+|x|^2), \tag{B25}$$

for some $K > 0$, and for all $t \in [0,T]$, $x = (S,\delta) \in D$ (see Theorem 5.2.9 in Karatzas-Shreve (1988) [K-01]). From (B20), (B25) is clear. Note that $f_\alpha(T-t;\theta)$ is bounded and so is α_T, i.e.,$\left|\alpha_T(t)\right| \le M$, for some $M > 0$, for all $t \in [0,T]$. Since $b(t,S_t,\delta_t)$ is a linear-affine function of δ_t and from the boundedness of $\alpha_T(t)$, then we conclude (B24). By choosing $S_0 \in (0,\infty)$, the process S_t can be written as

$$S_t = S_0\exp\left(\int_0^t \sqrt{\beta_1\delta_s + \beta_2}\,dW_s^{(1)} + \int_0^t\left(r - \delta_s + \lambda_S(\beta_1\delta_s + \beta_2) - \tfrac{1}{2}(\beta_1\delta_s + \beta_2)\right)ds\right) \tag{B26}$$

for all $t \in [0,T]$. Since δ_t never explodes or leaves $(-\frac{\beta_2}{\beta_1},\infty)$ before T, $\mathbb{Q}$-a.s. and, by (B26), S_t never explodes or leaves $(0,\infty)$ before T, $\mathbb{Q}$-a.s..

Combining the results obtained from Parts 1-2, we have, for given$(S_0,\delta_0) \in D$, there exists a unique strong solution $X \equiv (S_t,\delta_t)_{t\in[0,T]}$ of the SDEs (1.2.1) with the initial condition $X_0 = (S_0,\delta_0)$ and X never explodes or leaves D before T, $\mathbb{Q}$- a.s.. □

Appendix C

Calculation of the integral term

The integral term contained in Equation (1.2.28) can be expressed as

$$\int_0^t e^{-(\lambda_\delta \beta_1 - \kappa)s}(\lambda_\delta \beta_2 + \alpha_T(s))ds = f^{(0)}(t) + \sum_{k=1}^{K^\alpha} \alpha_k^{(1)} \left[f_k^{(1)}(s) \right]_{s=0}^{s=t} + \alpha_k^{(2)} \left[f_k^{(2)}(s) \right]_{s=0}^{s=t}, \qquad \text{(C01)}$$

where

$$f^{(0)}(t) := (\alpha_0 + \lambda_\delta \beta_2) \int_0^t e^{-(\lambda_\delta \beta_1 - \kappa)s} ds, \qquad \text{(C02)}$$

$$f_k^{(1)}(s) := \int f_\alpha(T-s)\cos(2\pi ks)ds, \qquad \text{(C03)}$$

$$f_k^{(2)}(s) := \int f_\alpha(T-s)\sin(2\pi ks)ds, \quad k = 1, \ldots, K^\alpha, \qquad \text{(C04)}$$

and

$$f_\alpha(T-s) = \frac{\sqrt{p} + p_2 + (\sqrt{p} - p_2)e^{\sqrt{p}(T-s)}}{e^{\sqrt{p}(T-s)}}. \qquad \text{(C05)}$$

The constants in (C05) are given by

$$p = p_2^2 + 4p_1 - 4p_1 p_3, \;\; p_1 = \tfrac{1}{2}\sigma_\delta^2 \beta_1, \;\; p_2 = (\lambda_\delta + \rho\sigma_\delta)\beta_1 - \kappa, \text{ and } p_3 = \lambda_S \beta_1.$$

Under this consideration, we have $\sigma_\delta = \rho = 0$, hence,

$$p = d^2, \sqrt{p} + p_2 = |d| - d, \text{ and } \sqrt{p} - p_2 = |d| + d, \qquad \text{(C06)}$$

where $d = \kappa - \beta_1 \lambda_\delta$. (C06) implies that

$$f_\alpha(T-s) = \begin{cases} -2de^{d(T-s)} & ; \quad d \le 0 \\ \\ 2d & ; \quad d > 0 \end{cases}. \qquad \text{(C07)}$$

Substituting $f_\alpha(T-s)$ in (C03) and (C04) with (C07) gives us the formulas of $f_k^{(1)}$ and $f_k^{(2)}$ as written within the proof of Proposition 2. Note that, if $d = 0$ then, from (C02), we have $f^{(0)}(t) = (\alpha_0 + \lambda_\delta \beta_2)t$ and $f_\alpha(T-s) = 0$. This implies, from Equation (1.2.28),

$$\overline{\delta}(t) = \overline{\delta}_0 + (\alpha_0 + \lambda_\delta \beta_2)t.$$

The just obtained solution is not preferred because it does not contain any term which can describe the seasonal behavior of the convenience yields.

Appendix D

Proof of Proposition 5

To avoid confusion about the notations, we omit writing the subscript t of S_t and δ_t in this proof unless they are necessary. The futures price $F^T \equiv F^T(t,S,\delta)$ satisfies

$$\frac{\partial F^T}{\partial t}+\frac{1}{2}(\beta_1\delta+\beta_2)S^2\frac{\partial^2 F^T}{\partial S^2}+\frac{1}{2}\sigma_\delta^2(\beta_1\delta+\beta_2)\frac{\partial^2 F^T}{\partial \delta^2}+\rho\sigma_\delta(\beta_1\delta+\beta_2)S\frac{\partial^2 F^T}{\partial\delta\partial S}$$

$$+(r-\delta+\lambda_S(\beta_1\delta+\beta_2))S\frac{\partial F^T}{\partial S}+(\alpha_T(t)-\kappa\delta+\lambda_\delta(\beta_1\delta+\beta_2))\frac{\partial F^T}{\partial\delta}=0, \quad \text{(D01)}$$

in U_T, subject to the terminal condition

$$F^T(T,S,\delta)=S \text{ in } (0,\infty)\times(\tfrac{-\beta_2}{\beta_1},\infty). \quad \text{(D02)}$$

Suppose that $F^T(t,S,\delta)$ is of the following form

$$F^T(t,S,\delta)=S\,e^{A(T-t;\theta)+B(T-t;\theta)\delta}, \quad \text{(D03)}$$

where θ is a vector of the unknown parameters, $A(T-t;\theta)$ and $B(T-t;\theta)$ are functions of time, independent of S and δ, to be determined.
Let $\tau=T-t$ and we calculate

$$\frac{\partial F^T}{\partial t}=-(A'(\tau;\theta)+B'(\tau;\theta)\delta)F^T,\quad \frac{\partial F^T}{\partial S}=\frac{F^T}{S},\quad \frac{\partial^2 F^T}{\partial S^2}=0,$$

$$\frac{\partial F^T}{\partial \delta}=B(\tau;\theta)F^T,\quad \frac{\partial^2 F^T}{\partial \delta^2}=B^2(\tau;\theta)F^T,\quad \frac{\partial^2 F^T}{\partial\delta\partial S}=\frac{B(\tau;\theta)}{S}F^T,$$

where $'=\dfrac{d}{d\tau}$.
Replacing the partial derivatives of F^T in (D01) with the above results, we then obtain the following equation

$$-(A'(\tau;\theta)+B'(\tau;\theta)\delta)+\frac{1}{2}\sigma_\delta^2(\beta_1\delta+\beta_2)B^2(\tau;\theta)+\rho\,\sigma_\delta(\beta_1\delta+\beta_2)B(\tau;\theta)$$

$$+(r-\delta+\lambda_S(\beta_1\delta+\beta_2))+(\alpha_T(T-\tau)-\kappa\delta+\lambda_\delta(\beta_1\delta+\beta_2))B(\tau;\theta)=0, \quad \text{(D04)}$$

which can be reduced to two ODEs by matching the coefficients of δ between the RHS and the LHS of (D04),

$$-B'(\tau;\theta) + p_1 B^2(\tau;\theta) + p_2 B(\tau;\theta) + p_3 - 1 = 0, \tag{D05}$$

$$-A'(\tau;\theta) + q_1 B^2(\tau;\theta) + (\alpha_T (T-\tau) + q_2) B(\tau;\theta) + q_3 = 0, \tag{D06}$$

where

$$p_1 = \frac{1}{2}\sigma_\delta^2 \beta_1, \quad p_2 = (\lambda_\delta + \rho\sigma_\delta)\beta_1 - \kappa, \quad p_3 = \lambda_S \beta_1,$$

$$q_1 = \frac{1}{2}\sigma_\delta^2 \beta_2, \quad q_2 = (\lambda_\delta + \rho\sigma_\delta)\beta_2, \quad q_3 = r + \lambda_S \beta_2.$$

The terminal condition in (D02) implies that

$$B(0;\theta) = 0, \tag{D07}$$

$$A(0;\theta) = 0, \tag{D08}$$

which are the initial conditions for solving the ODEs (D05) and (D06).

It should be remarked here from (D04) that we have only (D06) in the case that δ is equal to zero. This implies we have arbitrary choices for B. However, for the continuity of F^T in variable δ, we must choose B which satisfies (D05) and (D07).

The general solution of the Riccati ODE (D05) is of the following form:

$$B(\tau;\theta) = \frac{-\sqrt{p} - p_2}{2p_1} + \frac{1}{p_1\left(\dfrac{1}{\sqrt{p}} + e^{\sqrt{p}\tau} C_B\right)}, \tag{D09}$$

where $p = p_2^2 + 4p_1 - 4p_1 p_3$ and C_B is an arbitrary constant to be determined. Applying the initial condition (D07) to (D09), we get

$$C_B = \frac{\sqrt{p} - p_2}{\sqrt{p}(\sqrt{p} + p_2)}. \tag{D10}$$

Plugging C_B in (D10) into (D09) and simplifying the result, we then obtain the particular solution of (D05)

$$B(\tau;\theta) = -\frac{2(1-p_3)\left(e^{\sqrt{p}\tau} - 1\right)}{\left(\sqrt{p} + p_2 + (\sqrt{p} - p_2)e^{\sqrt{p}\tau}\right)}. \tag{D11}$$

Let

$$D(\tau;\theta) := q_3 + (\alpha_T (T-\tau) + q_2) B(\tau;\theta) + q_1 B^2(\tau;\theta), \tag{D12}$$

The general solution of (D06) can be written as

$$A(\tau;\theta) = C_A + \int_0^{\tau} D(s;\theta)ds, \tag{D13}$$

where C_A is an arbitrary constant to be determined. Applying the initial condition (D08) to (D13), we then have $C_A = 0$ and the particular solution of (D06) is obtained

$$A(\tau;\theta) = \int_0^{\tau} D(s;\theta)ds. \tag{D14}$$

We next verify the integral term in (D14). The integral term can be written as

$$\int_0^{\tau} D(s;\theta)ds = \int_0^{\tau} q_3\, ds + \int_0^{\tau} (q_2 + \alpha_T(T-s))B(s;\theta)ds + q_1 \int_0^{\tau} B^2(s;\theta)ds$$

$$= q_3\tau + (q_2 + \alpha_0)\int_0^{\tau} B(s;\theta)ds + q_1 \int_0^{\tau} B^2(s;\theta)ds$$

$$+\sum_{k=1}^{K^{\alpha}}\left(\alpha_k^{(1)} \int_0^{\tau} \cos(2\pi k(T-s)) f_{\alpha}(s;\theta)B(s;\theta)ds + \alpha_k^{(2)} \int_0^{\tau} \sin(2\pi k(T-s)) f_{\alpha}(s;\theta)B(s;\theta)ds\right). \tag{D15}$$

Let $f_1, f_2, f_c,$ and f_s are functions which satisfy

$$f_1(s;\theta) = \int B(s;\theta)ds, \tag{D16}$$

$$f_2(s;\theta) = \int B^2(s;\theta)ds, \tag{D17}$$

$$f_c(s;\theta) = \int \cos(2\pi k(T-s)) f_{\alpha}(s;\theta) B(s;\theta)\, ds, \tag{D18}$$

$$f_s(s;\theta) = \int \sin(2\pi k(T-s)) f_{\alpha}(s;\theta) B(s;\theta)\, ds. \tag{D19}$$

Applying (D16)-(D19) to (D15), we can express $A(\tau;\theta)$ in the following form:

$$A(\tau;\theta) = (r + \lambda_S \beta_2)\tau + \left(q_2 + \alpha_0\right)\Big[f_1(s;\theta)\Big]_{s=0}^{s=\tau} + q_1\left[f_2(s;\theta)\right]_{s=0}^{s=\tau}$$

$$+\sum_{k=1}^{K^{\alpha}}\left(\alpha_k^{(1)}\left[f_c(s,T,k;\theta)\right]_{s=0}^{s=\tau} + \alpha_k^{(2)}\left[f_s(s,T,k;\theta)\right]_{s=0}^{s=\tau}\right). \tag{D20}$$

To verify the integral terms in (D16)-(D19), we need the following identities:

$$\int \frac{e^{cs}-1}{a+be^{cs}}\,ds = -\frac{s}{a} + \frac{(a+b)\ln\left(a+be^{cs}\right)}{abc}, \tag{D21}$$

$$\int \left(\frac{e^{cs}-1}{a+be^{cs}}\right)^2 ds = \frac{s}{a^2} + \frac{(a+b)^2}{ab^2c\left(a+be^{cs}\right)} + \frac{(a^2-b^2)\ln\left(a+be^{cs}\right)}{a^2b^2c}, \tag{D22}$$

$$\int \cos(m(T-s))\left(\frac{e^{cs}-1}{e^{cs}}\right) ds$$

$$= \frac{mc\cos(m(T-s)) - \left(m^2\left(e^{cs}-1\right)+c^2e^{cs}\right)\sin(m(T-s))}{m(m^2+c^2)e^{cs}}, \tag{D23}$$

$$\int \sin(m(T-s))\left(\frac{e^{cs}-1}{e^{cs}}\right) ds$$

$$= \frac{\left(m^2\left(e^{cs}-1\right)+c^2e^{cs}\right)\cos(m(T-s)) + mc\sin(m(T-s))}{m(m^2+c^2)e^{cs}}, \tag{D24}$$

where $a, b, c,$ and m are arbitrary constants which make (D21)-(D24) are well-defined. We have chosen

$$f_\alpha(t;\theta) = \frac{\sqrt{p}+p_2+(\sqrt{p}-p_2)e^{\sqrt{p}t}}{e^{\sqrt{p}t}}, \tag{D25}$$

for all $t \in [0,T]$ and then we have

$$\cos(2\pi k(T-s))f_\alpha(s;\theta)B(s;\theta) = -2(1-p_3)\cos(2\pi k(T-s))\left(\frac{e^{\sqrt{p}s}-1}{e^{\sqrt{p}s}}\right), \tag{D26}$$

$$\sin(2\pi k(T-s))f_\alpha(s;\theta)B(s;\theta) = -2(1-p_3)\sin(2\pi k(T-s))\left(\frac{e^{\sqrt{p}s}-1}{e^{\sqrt{p}s}}\right). \tag{D27}$$

By setting

$$a = \sqrt{p}+p_2,\ b = \sqrt{p}-p_2,\ c = \sqrt{p},\ \text{and}\ m = 2\pi k,$$

and using the identities (D21)-(D24) , we obtain

$$\int B(s;\theta)ds$$

$$= -2(1-p_3)\int \frac{e^{\sqrt{p}s}-1}{\sqrt{p}+p_2+(\sqrt{p}-p_2)e^{\sqrt{p}s}}ds$$

$$= -2(1-p_3)\left(-\frac{s}{\sqrt{p}+p_2}+\frac{((\sqrt{p}+p_2)+(\sqrt{p}-p_2))}{(\sqrt{p}+p_2)(\sqrt{p}-p_2)\sqrt{p}}\ln\left(\sqrt{p}+p_2+(\sqrt{p}-p_2)e^{\sqrt{p}s}\right)\right)$$

$$= \underbrace{\frac{(\sqrt{p}-p_2)s-2\ln\left(\sqrt{p}+p+(\sqrt{p}-p_2)e^{\sqrt{p}s}\right)}{2p_1}}_{f_1(s;\theta)}, \tag{D28}$$

$$\int B^2(s;\theta)ds$$

$$= 4(1-p_3)^2\int\left(\frac{e^{\sqrt{p}s}-1}{\sqrt{p}+p_2+(\sqrt{p}-p_2)e^{\sqrt{p}s}}\right)^2 ds$$

$$= \left(4(1-p_3)^2\right)\left(\frac{s}{(\sqrt{p}+p_2)^2}+\frac{((\sqrt{p}+p_2)+(\sqrt{p}-p_2))^2}{(\sqrt{p}+p_2)(\sqrt{p}-p_2)^2\sqrt{p}\left(\sqrt{p}+p_2+(\sqrt{p}-p_2)e^{\sqrt{p}s}\right)}\right.$$

$$\left.+\frac{((\sqrt{p}+p_2)^2-(\sqrt{p}-p_2)^2)}{(\sqrt{p}+p_2)^2(\sqrt{p}-p_2)^2\sqrt{p}}\ln\left(\sqrt{p}+p_2+(\sqrt{p}-p_2)e^{\sqrt{p}s}\right)\right)$$

$$= \underbrace{\frac{1}{4p_1^2}\left((\sqrt{p}-p_2)^2 s+\frac{4\sqrt{p}\,(\sqrt{p}+p_2)}{\left(\sqrt{p}+p_2+(\sqrt{p}-p_2)e^{\sqrt{p}s}\right)}+4p_2\ln\left(\sqrt{p}+p_2+(\sqrt{p}-p_2)e^{\sqrt{p}s}\right)\right)}_{f_2(s;\theta)} \tag{D29}$$

$$\int \cos(2\pi k(T-s))f_\alpha(s;\theta)B(s;\theta)ds$$

$$= -2(1-p_3)\int\cos(2\pi k(T-s))\left(\frac{e^{\sqrt{p}s}-1}{e^{\sqrt{p}s}}\right)ds \qquad \text{(use (D26))}$$

$$= \underbrace{-\frac{(1-p_3)\left(2\pi k\sqrt{p}\cos(2\pi k(T-s))-\left((2\pi k)^2\left(e^{\sqrt{p}s}-1\right)+p\,e^{\sqrt{p}s}\right)\sin(2\pi k(T-s))\right)}{\pi k\left((2\pi k)^2+p\right)e^{\sqrt{p}s}}}_{f_c(s,T,k;\theta)} \tag{D30}$$

$$\int \sin(2\pi k(T-s)) f_\alpha(s;\theta) B(s;\theta) ds$$

$$= -2(1-p_3)\int \sin(2\pi k(T-s))\left(\frac{e^{\sqrt{p}s}-1}{e^{\sqrt{p}s}}\right) ds \qquad \text{(use (D27))}$$

$$= \underbrace{-\frac{(1-p_3)\left(\left((2\pi k)^2\left(e^{\sqrt{p}s}-1\right)+pe^{\sqrt{p}s}\right)\cos(2\pi k(T-s))+2\pi k\sqrt{p}\sin(2\pi k(T-s))\right)}{\pi k((2\pi k)^2+p)e^{\sqrt{p}s}}}_{f_s(s,T,k;\theta)}.$$

(D31)

Applying (D28)-(D31) to (D16)-(D19), respectively, we obtain $f_1, f_2, f_c,$ and f_s as written in Proposition 5.

We next consider the differentiability of $F^T(\tau,S,\delta;\theta)$. It is obvious that the differentiability of $F^T(\tau,S,\delta;\theta)$ depends on the differentiability of $B(\tau;\theta)$, $f_1(\tau;\theta)$, $f_2(\tau;\theta)$, $f_c(\tau,T,k;\theta)$, and $f_s(\tau,T,k;\theta)$ in variable τ on $[0,T]$.
Consider

$$B(\tau;\theta) = -\frac{2(1-p_3)\left(e^{\sqrt{p}\tau}-1\right)}{\left(\sqrt{p}+p_2+(\sqrt{p}-p_2)e^{\sqrt{p}\tau}\right)}, \tau \geq 0. \tag{D32}$$

Thus, we must have

$$p > 0 \tag{D33}$$

and

$$Q(\tau) := \left(\sqrt{p}+p_2+(\sqrt{p}-p_2)e^{\sqrt{p}\tau}\right) \neq 0 \text{ for all } \tau \geq 0, \tag{D34}$$

for the differentiability of $B(\tau;\theta)$. We have $Q(\tau)=0$ if and only if

$$\tau = \frac{\ln\left(-\left(\dfrac{\sqrt{p}+p_2}{\sqrt{p}-p_2}\right)\right)}{\sqrt{p}}. \tag{D35}$$

To make $Q(\tau) \neq 0$ for all $\tau \geq 0$, we assume that

$$p_2 \neq \pm\sqrt{p} \tag{D36}$$

and

$$\frac{\sqrt{p}+p_2}{-\sqrt{p}+p_2} < 0. \quad \text{(D37)}$$

The conditions (D36) and (D37) imply that

$$|p_2| < \sqrt{p}. \quad \text{(D38)}$$

The condition (D38) implies

$$\sqrt{p}+p_2 > 0 \text{ and } \sqrt{p}-p_2 > 0. \quad \text{(D39)}$$

The conditions (D33) and (D38) are also the sufficient conditions for differentiability of $f_1(\tau;\theta)$, $f_2(\tau;\theta)$, $f_c(\tau,T,k;\theta)$, and $f_s(\tau,T,k;\theta)$ in variable τ on $[0,T]$.

It should be noted that the condition (D38) implies that

$$4p_1(1-p_3) = p - p_2^2 > 0. \quad \text{(D40)}$$

Since we have $p_1 = \frac{1}{2}\sigma_\delta^2\beta_1 > 0$ and (D40), $(1-p_3)$ must be positive.

Finally, we show some properties of $f_\alpha(T-t;\theta)$ for $t \in [0,T]$. The conditions in (D39) implies that $f_\alpha(T-t;\theta)$ is a strictly increasing function in *t*, and hence,

$$\inf_{t\in[0,T]} f_\alpha(T-t;\theta) = f_\alpha(T;\theta) = \frac{\sqrt{p}+p_2+(\sqrt{p}-p_2)e^{\sqrt{p}T}}{e^{\sqrt{p}T}} \geq \sqrt{p}-p_2 > 0, \quad \text{(D41)}$$

$$\sup_{t\in[0,T]} f_\alpha(T-t;\theta) = f_\alpha(0;\theta) = 2\sqrt{p}. \quad \text{(D42)}$$

From (D41)-(D42), we have $f_\alpha(T-t;\theta)$ varies within the range $[\sqrt{p}-p_2, 2\sqrt{p}]$ for all $t \in [0,T]$. Since $\sqrt{p}-p_2 > 0$, $f_\alpha(T-t;\theta) > 0$ for all $t \in [0,T]$. □

Appendix E

Proof of Proposition 6

Proof (1).

We have $p > 0$ and $|p_2| < \sqrt{p}$ which imply

$$\sqrt{p} + p_2 > 0 \text{ and } \sqrt{p} - p_2 > 0. \tag{E01}$$

Since $(1 - p_3) > 0$ and $e^{\sqrt{p}\tau} \geq 1$ for all $\tau \geq 0$, we have

$$B(\tau;\theta) = -\frac{2(1-p_3)\left(e^{\sqrt{p}\tau} - 1\right)}{\left(\sqrt{p} + p_2 + (\sqrt{p} - p_2)e^{\sqrt{p}\tau}\right)} \leq 0, \tag{E02}$$

for all $\tau \geq 0$.

Differentiating $B(\tau;\theta)$ with respect to τ, one obtains

$$B'(\tau;\theta) = -\frac{4(1-p_3)p\, e^{\sqrt{p}\tau}}{\left(\sqrt{p} + p_2 + (\sqrt{p} - p_2)e^{\sqrt{p}\tau}\right)^2} < 0, \tag{E03}$$

for all $\tau \geq 0$.

(E03) implies that

$$B(\tau;\theta) \text{ is strictly decreasing on } [0,\infty). \tag{E04}$$

Since $B(0;\theta) = 0$ and from (E04), we have

$$|B(\tau;\theta)| \text{ is strictly increasing on } [0,\infty). \tag{E05}$$

Next, we calculate

$$\lim_{\tau'\to\infty} |B(\tau';\theta)| = \lim_{\tau'\to\infty} \left| -\frac{2(1-p_3)\left(e^{\sqrt{p}\tau'} - 1\right)}{\left(\sqrt{p} + p_2 + (\sqrt{p} - p_2)\, e^{\sqrt{p}\tau'}\right)} \right| = \frac{2(1-p_3)}{\sqrt{p} - p_2}. \tag{E06}$$

(E05) and (E06) imply that the following estimate holds

$$|B(\tau;\theta)| \leq \lim_{\tau'\to\infty} |B(\tau';\theta)| = \frac{2(1-p_3)}{\sqrt{p} - p_2}, \tag{E07}$$

for all $\tau \geq 0$.

From Appendix D, we have

$$\frac{\partial F^T}{\partial \delta} = B(\tau;\theta)F^T < 0 \quad \text{and} \quad \frac{\partial^2 F^T}{\partial \delta^2} = B^2(\tau;\theta)F^T > 0,$$

for all $(\tau, S, \delta) \in U_T$. These results imply that the mapping $\delta \mapsto F^T(t,S,\delta)$ is strictly decreasing and strictly convex on $(\frac{-\beta_2}{\beta_1}, \infty)$. Since B is negative on $(0,T)$ and by the formula (1.3.6), we have $\lim\limits_{\delta\to\infty} F^T(t,S,\delta) = 0$. □

Proof (2).
We have

$$f_1(\tau;\theta) = \int_0^\tau B(s;\theta)ds + f_1(0;\theta) \quad \text{and} \quad f_2(\tau;\theta) = \int_0^\tau B^2(s;\theta)ds + f_2(0;\theta).$$

Since $B(\tau;\theta)$ is nonpositive strictly decreasing on $[0,\infty)$ and $B(0;\theta) = 0$, we must have $\left[f_1(s;\theta)\right]_{s=0}^{s=\tau}$ is nonpositive strictly decreasing on $[0,\infty)$ and $\left[f_2(s;\theta)\right]_{s=0}^{s=\tau}$ is nonnegative strictly increasing on $[0,\infty)$. □

Proof (3).
We have $-1 \le \sin(\omega), \cos(\omega) \le 1$ for all $\omega \in \mathbb{R}$ and one can show that $e^{\sqrt{p}\tau} - 1 \ge 0$ for all $\tau \ge 0$. Hence, we have the following estimate:

$$\left|f_c(s,T,k;\theta)\right| = \left| -\frac{(1-p_3)\left(2\pi k\sqrt{p}\cos(2\pi k(T-s)) - \left((2\pi k)^2\left(e^{\sqrt{p}s}-1\right) + pe^{\sqrt{p}s}\right)\sin(2\pi k(T-s))\right)}{\pi k\left((2\pi k)^2+p\right)e^{\sqrt{p}s}} \right|$$

$$\le (1-p_3)\frac{\left(2\pi k\sqrt{p} + (2\pi k)^2\left(e^{\sqrt{p}\tau}-1\right) + pe^{\sqrt{p}\tau}\right)}{\pi k\left((2\pi k)^2+p\right)e^{\sqrt{p}\tau}}$$

$$\le (1-p_3)\frac{\left(2\pi k\sqrt{p} + (2\pi k)^2 + p\right)}{\pi k\left((2\pi k)^2+p\right)}$$

$$\le (1-p_3)\left(\frac{2\sqrt{p}}{(2\pi k)^2+p} + \frac{1}{\pi k}\right)$$

$$\le (1-p_3)\left(\frac{2\sqrt{p}}{(2\pi)^2+p} + \frac{1}{\pi}\right),$$

for all $\tau > 0$ and for all $k = 1,2,\ldots,K^\alpha$. The above estimate also holds for $f_s(s,T,k;\theta)$. □

Appendix F

Proof of Proposition 7 and Proposition 8

Proof (Proposition 7).

By analogy to the Black-Scholes formula, we suppose that the solution of the PDE (1.3.25) is of the following form:

$$C(t, F_t^T, \delta_t; T_c, T, K; \theta) = e^{-r(T_c - t)}\left(F_t^T P_1 - K P_2\right), \tag{F01}$$

where $x_t = \ln F_t^T$ and $P_j \equiv P_j(t, x_t, \delta_t; T_c, T, \ln K; \theta)$, $j = 1, 2$, are functions to be determined. To avoid confusion with the notations, we omit the superscript T and the subscript t of $F_t^T, \delta_t,$ and x_t. We next calculate the following partial derivatives:

$$\frac{\partial C}{\partial t} = e^{-r(T_c - t)}\left(F\left(\frac{\partial P_1}{\partial t} + rP_1\right) - K\left(\frac{\partial P_2}{\partial t} + rP_2\right)\right),$$

$$\frac{\partial C}{\partial F} = \frac{\partial C}{\partial x}\frac{\partial x}{\partial F} = e^{-r(T_c - t)}\left(\left(\frac{\partial P_1}{\partial x} + P_1\right) - \frac{K}{F}\frac{\partial P_2}{\partial x}\right),$$

$$\frac{\partial^2 C}{\partial F^2} = e^{-r(T_c - t)}\left(\frac{1}{F}\left(\frac{\partial^2 P_1}{\partial x^2} + \frac{\partial P_1}{\partial x}\right) - \frac{K}{F^2}\left(\frac{\partial^2 P_2}{\partial x^2} - \frac{\partial P_2}{\partial x}\right)\right),$$

$$\frac{\partial C}{\partial \delta} = e^{-r(T_c - t)}\left(F\frac{\partial P_1}{\partial \delta} - K\frac{\partial P_2}{\partial \delta}\right), \quad \frac{\partial^2 C}{\partial \delta \partial F} = e^{-r(T_c - t)}\left(\left(\frac{\partial^2 P_1}{\partial \delta \partial x} + \frac{\partial P_1}{\partial \delta}\right) - \frac{K}{F}\frac{\partial^2 P_2}{\partial \delta \partial x}\right),$$

$$\frac{\partial^2 C}{\partial \delta^2} = e^{-r(T_c - t)}\left(F\frac{\partial^2 P_1}{\partial \delta^2} - K\frac{\partial^2 P_2}{\partial \delta^2}\right).$$

Plugging the above results into Equation (1.3.25) gives us the PDE:

$$\begin{aligned}
&\left(F\left(\frac{\partial P_1}{\partial t} + rP_1\right) - K\left(\frac{\partial P_2}{\partial t} + rP_2\right)\right) + \tfrac{1}{2}\sigma_F^2(\cdot)F^2\left(\frac{1}{F}\left(\frac{\partial^2 P_1}{\partial x^2} + \frac{\partial P_1}{\partial x}\right) - \frac{K}{F^2}\left(\frac{\partial^2 P_2}{\partial x^2} - \frac{\partial P_2}{\partial x}\right)\right) \\
&\quad + \tfrac{1}{2}\sigma_\delta^2\left(\beta_1\delta + \beta_2\right)\left(F\frac{\partial^2 P_1}{\partial \delta^2} - K\frac{\partial^2 P_2}{\partial \delta^2}\right) + \sigma_{F\delta}(\cdot)F\left(\left(\frac{\partial^2 P_1}{\partial \delta \partial x} + \frac{\partial P_1}{\partial \delta}\right) - \frac{K}{F}\frac{\partial^2 P_2}{\partial \delta \partial x}\right) \\
&\quad + \mu_\delta(\cdot)\left(F\frac{\partial P_1}{\partial \delta} - K\frac{\partial P_2}{\partial \delta}\right) - r\left(FP_1 - KP_2\right) = 0.
\end{aligned} \tag{F02}$$

We can separate the LHS of (F02) into two parts: one contains only P_1 and its partial derivatives and the other one contains only P_2 and its partial derivatives. Equating each part to zero gives us the following PDEs:

$$\frac{\partial P_j}{\partial t} + \tfrac{1}{2}\sigma_F^2(\cdot)\frac{\partial^2 P_j}{\partial x^2} + \tfrac{1}{2}\sigma_\delta^2\left(\beta_1\delta + \beta_2\right)\frac{\partial^2 P_j}{\partial \delta^2} + \sigma_{F\delta}(\cdot)\frac{\partial^2 P_j}{\partial \delta \partial x} - \tfrac{(-1)^j}{2}\sigma_F^2(\cdot)\frac{\partial P_j}{\partial x} + \Big((2-j)\sigma_{F\delta}(\cdot) + \mu_\delta(\cdot)\Big)\frac{\partial P_j}{\partial \delta} = 0, \tag{F03}$$

for $j = 1,2,$ where the functions in (F03) are given by

$\sigma_F^2(s,\delta;\theta) = \left(\beta_1\delta + \beta_2\right)\left(1 + 2\rho\sigma_\delta B(s;\theta) + \sigma_\delta^2 B^2(s;\theta)\right),$

$\sigma_{F\delta}(s,\delta;\theta) = \left(\beta_1\delta + \beta_2\right)\left(\rho\sigma_\delta + \sigma_\delta^2 B(s;\theta)\right),$

$\mu_\delta(s,\delta;\theta) = \alpha_T(s) - \kappa\delta + \lambda_\delta\left(\beta_1\delta + \beta_2\right),$ for $s \geq 0.$

In order to satisfy the terminal condition in Equation (1.3.26), P_j must satisfy the terminal conditions:

$$P_j(T_c, x, \delta; T_c, T, \ln K; \theta) = 1_{\{x \geq \ln K\}}, \quad j = 1,2. \tag{F04}$$

Next, we will show that $P_j, j = 1,2,$ are conditional probabilities. For each j, we consider the following Itô diffusion processes:

$$\begin{aligned} dx_t^{(j)} &= -\tfrac{(-1)^j}{2}\sigma_F^2(T-t,\delta_t^{(j)};\theta)\,dt + \sigma_F(T-t,\delta_t^{(j)};\theta)\,d\tilde{W}_t^{(1)}, \\ d\delta_t^{(j)} &= \Big((2-j)\,\sigma_{F\delta}(T-t,\delta_t^{(j)};\theta) + \mu_\delta(t,\delta_t^{(j)};\theta)\Big)\,dt \\ &\qquad + \sigma_\delta\sqrt{\beta_1\delta_t^{(j)} + \beta_2}\left(\tilde{\rho}(t,T;\theta)d\tilde{W}_t^{(1)} + \sqrt{1-\tilde{\rho}^2(t,T;\theta)}\,d\tilde{W}_t^{(2)}\right), \end{aligned} \tag{F05}$$

with a two-dimensional Brownian motion $\tilde{W} \equiv (\tilde{W}_t^{(1)}, \tilde{W}_t^{(2)})_{t\in[0,T]}$ where $\tilde{W}_t^{(1)}$ and $\tilde{W}_t^{(2)}$ are independent Brownian motions and the correlation function $\tilde{\rho}(t,T;\theta)$ is given by

$$\tilde{\rho}(t,T;\theta) := \frac{\rho + \sigma_\delta B(T-t;\theta)}{\sqrt{\zeta(t;\theta)}} = \frac{\rho + \sigma_\delta B(T-t;\theta)}{\sqrt{1 + 2\rho\sigma_\delta B(T-t;\theta) + \sigma_\delta^2 B^2(T-t;\theta)}}, \tag{F06}$$

where $\zeta(t;\theta)$ is defined in Corollary 1.3.1 and, under the additional condition (A4) imposed in the corollary, we have $\zeta(t;\theta) > 0.$ Note that $\left|\tilde{\rho}(t,T;\theta)\right| < 1$ for all $t \in [0,T].$

Let $\tilde{p}_j = \tilde{p}_j(T_c, x_T^{(j)}, \delta_T^{(j)}; t, x_t^{(j)}, \delta_t^{(j)}; T;\theta),$ $j = 1,2,$ be, respectively, the transition densities of the diffusion processes $(x_t^{(j)}, \delta_t^{(j)})_{t\in[0,T]}$. One can verify that, for each j, $\tilde{p}_j$ is the fundamental solution to the parabolic PDE governed by (F03), and we can express P_j as the following forms:

$$P_j(t,x,\delta;T_c,T,\ln K;\theta) = \int_{\mathcal{D}^{(j)}_{(x,\delta)}} 1_{\{x^{(j)}_{T_c} \geq \ln K\}} \tilde{p}_j(T_c, x^{(j)}_{T_c}, \delta^{(j)}_{T_c}; t, x, \delta; T;\theta)\, dx^{(j)}_{T_c} d\delta^{(j)}_{T_c}$$

$$= \Pr\Big[x^{(j)}_{T_c} \geq \ln K \mid x^{(j)}_t = x, \delta^{(j)}_t = \delta; T;\theta\Big], \tag{F07}$$

for $j = 1,2$, where $\mathcal{D}^{(j)}_{(x,\delta)}$ denotes the domain of the diffusion process $(x^{(j)}_t, \delta^{(j)}_t)$. From (F07), we have shown that P_j, $j = 1,2$, are, in fact, the conditional probabilities.

Let

$$\varphi_j(\phi;t,x,\delta;T_c,T;\theta) = E\Big[e^{\mathbb{i}\phi x^{(j)}_{T_c}} \mid x^{(j)}_t = x, \delta^{(j)}_t = \delta\Big] \tag{F08}$$

be the characteristic function of $x^{(j)}_{T_c}$ for $j = 1,2$. In fact, φ_j is the Fourier transform of the random variable $x^{(j)}_{T_c}$ for $j = 1,2$. We recall here one property of the characteristic functions:

$$\Pr\Big[x^{(j)}_{T_c} < \ln K \mid x^{(j)}_t = x, \delta^{(j)}_t = \delta; T;\theta\Big] = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \operatorname{Re}\left(\frac{e^{-\mathbb{i}\phi \ln K}\varphi_j(\phi;t,x,\delta;T_c,T;\theta)}{\mathbb{i}\phi}\right) d\phi.$$

Applying the above property to (F07), we obtain

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \operatorname{Re}\left[\frac{e^{-\mathbb{i}\phi \ln K}\varphi_j(\phi;t,x,\delta;T_c,T;\theta)}{\mathbb{i}\phi}\right] d\phi,$$

for $j = 1,2$, as written in Equation (1.3.31). From Appendix in Heston (1993) [H-02], we have, for each j, φ_j satisfies the PDE (F03), i.e.,

$$\frac{\partial \varphi_j}{\partial t} + ½\sigma_F^2(\cdot)\frac{\partial^2 \varphi_j}{\partial x^2} + ½\sigma_\delta^2\left(\beta_1\delta + \beta_2\right)\frac{\partial^2 \varphi_j}{\partial \delta^2} + \sigma_{F\delta}(\cdot)\frac{\partial^2 \varphi_j}{\partial \delta \partial x} - \tfrac{(-1)^j}{2}\sigma_F^2(\cdot)\frac{\partial \varphi_j}{\partial x}$$
$$+\Big((2-j)\sigma_{F\delta}(\cdot) + \mu_\delta(\cdot)\Big)\frac{\partial \varphi_j}{\partial \delta} = 0, \tag{F09}$$

subject to the terminal condition

$$\varphi_j(\phi;T_c,x,\delta;T_c,T;\theta) = e^{\mathbb{i}\phi x}. \tag{F10}$$

To solve the PDE (F09) subject to the terminal condition (F10), we suppose that φ_j can be expressed as in the following form:

$$\varphi_j(\phi;t,x,\delta;T_c,T;\theta)=e^{\tilde{A}_j(\phi;T_c-t,T_c,T;\theta)+\tilde{B}_j(\phi;T_c-t,T_c,T;\theta)\,\delta+\mathbb{i}\phi x}, \tag{F11}$$

where $\tilde{A}_j$ and $\tilde{B}_j, j=1,2,$ are functions to be determined.

Let $\tau=T_c-t$. We next calculate the following partial derivatives:

$$\frac{\partial\varphi_j}{\partial t}=-\left(\tilde{A}_j'(\phi;\tau,T_c,T;\theta)+\tilde{B}_j'(\phi;\tau,T_c,T;\theta)\delta\right)\varphi_j,\quad \frac{\partial\varphi_j}{\partial x}=\mathbb{i}\phi\varphi_j,$$

$$\frac{\partial^2\varphi_j}{\partial x^2}=-\phi^2\varphi_j,\ \frac{\partial\varphi_j}{\partial\delta}=\tilde{B}_j(\phi;\tau,T_c,T;\theta)\varphi_j,\ \frac{\partial^2\varphi_j}{\partial\delta\partial x}=\mathbb{i}\phi\,\tilde{B}_j(\phi;\tau,T_c,T;\theta)\varphi_j,$$

$$\frac{\partial^2\varphi_j}{\partial\delta^2}=\tilde{B}_j^2(\phi;\tau,T_c,T;\theta)\varphi_j,\quad j=1,2,\ \text{ where }\ '=\frac{d}{d\tau}.$$

Putting the above partial derivatives into (F09) gives us the following ODEs:

$$\begin{aligned}
&-\left(\tilde{A}_j'(\phi;\tau,T_c,T;\theta)+\tilde{B}_j'(\phi;\tau,T_c,T;\theta)\delta\right)-\tfrac{1}{2}\sigma_F^2(T-t,\delta;\theta)\phi^2\\
&\qquad+\tfrac{1}{2}\sigma_\delta^2(\beta_1\delta+\beta_2)\tilde{B}_j^2(\phi;\tau,T_c,T;\theta)+\sigma_{F\delta}(T-t,\delta;\theta)\mathbb{i}\phi\tilde{B}_j(\phi;\tau,T_c,T;\theta)\\
&\qquad-\tfrac{(-1)^j}{2}\sigma_F^2(T-t;\delta;\theta)\mathbb{i}\phi+((2-j)\sigma_{F\delta}(T-t,\delta;\theta)+\mu_\delta(t,\delta;\theta)\tilde{B}_j(\phi;\tau,T_c,T;\theta)=0,
\end{aligned} \tag{F12}$$

for $j=1,2$.

For each j, we match the coefficients of δ between the RHS and the LHS of (F12) and then we obtain the two systems of ODEs as written in Proposition 7.

The terminal condition (F10) implies that the initial conditions $\tilde{A}_j(\phi;0,T_c,T;\theta)=0$ and $\tilde{B}_j(\phi;0,T_c,T;\theta)=0$ must hold for $j=1,2$. □

Proof (Proposition 8).

We claim that the European put futures option prices can be expressed as

$$P(t,F^T,\delta;T_p,T,K;\theta)=e^{-r(T_p-t)}\left(K(1-P_2)-F^T(1-P_1)\right). \tag{F13}$$

Next, we have to show that: (Step 1) P satisfies the following PDE:

$$\begin{aligned}
\frac{\partial P}{\partial t}&+\frac{1}{2}\sigma_F^2(T-t,\delta;\theta)F^2\frac{\partial^2P}{\partial F^2}+\frac{1}{2}\sigma_\delta^2(\beta_1\delta+\beta_2)\frac{\partial^2P}{\partial\delta^2}\\
&+\sigma_{F\delta}(T-t,\delta;\theta)F\frac{\partial^2P}{\partial F\partial\delta}+\mu_\delta(t,\delta;\theta)\frac{\partial P}{\partial\delta}-rP=0,
\end{aligned} \tag{F14}$$

and (Step 2) P satisfies the terminal condition as written in Equation (1.3.35).

Step 1

Let us suppose that there exists a call futures option written on the commodity futures contract expired at date $T_c = T_p$. From (F13), we have the put-call parity (1.3.37), i.e.,

$$P = e^{-r(T_p - t)}(K - F^T) + C, \tag{F15}$$

for all $t \in [0, T_p]$, where C is the call futures option price expired at T_p.

We calculate the following partial derivatives:

$$\frac{\partial P}{\partial t} = re^{-r(T_p - t)}(K - F^T) + \frac{\partial C}{\partial t}, \frac{\partial P}{\partial F} = -e^{-r(T_p - t)} + \frac{\partial C}{\partial F},$$

$$\frac{\partial^2 P}{\partial F^2} = \frac{\partial^2 C}{\partial F^2}, \quad \frac{\partial P}{\partial \delta} = \frac{\partial C}{\partial \delta}, \quad \frac{\partial^2 P}{\partial \delta \partial F} = \frac{\partial^2 C}{\partial \delta \partial F}, \quad \frac{\partial^2 P}{\partial \delta^2} = \frac{\partial^2 C}{\partial \delta^2}.$$

Plugging the above partial derivatives into the LHS of (F14) and using Equation (1.3.25), we have

$$\left\{ \frac{\partial C}{\partial t} + \frac{1}{2}\sigma_F^2(\cdot)F^2 \frac{\partial^2 C}{\partial F^2} + \frac{1}{2}\sigma_\delta^2(\beta_1 \delta + \beta_2)\frac{\partial^2 C}{\partial \delta^2} + \sigma_{F\delta}(\cdot)F \frac{\partial^2 C}{\partial F \partial \delta} + \mu_\delta(\cdot)\frac{\partial C}{\partial \delta} - rC \right\}$$
$$+ \left\{ re^{-r(T_p - t)}(K - F^T) - re^{-r(T_p - t)}(K - F^T) \right\} = 0.$$

Thus, P satisfies the PDE (F14).

Step 2

We verify $P(t, F^T, \delta; T_p, T, K; \theta)$ at $t = T_p$:

$$\begin{aligned}
&P(T_p, F^T, \delta; T_p, T, K; \theta) \\
&\quad = K(1 - P_2(T_p, \ln F^T, \delta; T_p, T, \ln K; \theta)) - F^T(1 - P_1(T_p, \ln F^T, \delta; T_p, T, \ln K; \theta)) \\
&\quad = K\left(1 - 1_{\{\ln F^T \geq \ln K\}}\right) - F^T\left(1 - 1_{\{\ln F^T \geq \ln K\}}\right) \\
&\quad = 1_{\{\ln F^T < \ln K\}}\left(K - F^T\right) \\
&\quad = \begin{cases} K - F^T & \text{if} \quad \ln F^T < \ln K \\ 0 & \text{if} \quad \ln F^T \geq \ln K \end{cases} = \max(0, K - F^T).
\end{aligned}$$

Hence, P satisfies the terminal condition as written in Equation (1.3.35). □

Appendix G

Evaluation of Call Futures Option Prices

Let $\tilde{A}_j(\phi;\tau,T_c,T;\theta) = \tilde{A}_{1j}(\phi;\tau,T_c,T;\theta) + \tilde{A}_{2j}(\phi;\tau,T_c,T;\theta)\mathbb{i}$, (G01)

$$\tilde{B}_j(\phi;\tau,T_c,T;\theta) = \tilde{B}_{1j}(\phi;\tau,T_c,T;\theta) + \tilde{B}_{2j}(\phi;\tau,T_c,T;\theta)\mathbb{i}, \tag{G02}$$

$$\tilde{C}_{\tilde{A}_j}(\phi;\tau,T_c,T;\theta) = \tilde{C}_{1\tilde{A}_j}(\phi;\tau,T_c,T;\theta) + \tilde{C}_{2\tilde{A}_j}(\phi;\tau,T_c,T;\theta)\mathbb{i}, \tag{G03}$$

$$\tilde{C}_{\tilde{B}_j}(\phi;\tau,T_c,T;\theta) = \tilde{C}_{1\tilde{B}_j}(\phi;\tau,T_c,T;\theta) + \tilde{C}_{2\tilde{B}_j}(\phi;\tau,T_c,T;\theta)\mathbb{i}, \tag{G04}$$

$$\tilde{D}_{\tilde{A}_j}(\phi;\tau,T_c,T;\theta) = \tilde{D}_{1\tilde{A}_j}(\phi;\tau,T_c,T;\theta) + \tilde{D}_{2\tilde{A}_j}(\phi;\tau,T_c,T;\theta)\mathbb{i}, \tag{G05}$$

$$\tilde{D}_{\tilde{B}_j}(\phi;\tau,T_c,T;\theta) = \tilde{D}_{1\tilde{B}_j}(\phi;\tau,T_c,T;\theta) + \tilde{D}_{2\tilde{B}_j}(\phi;\tau,T_c,T;\theta)\mathbb{i}, \tag{G06}$$

for $j = 1,2,$ where all functions on the RHS of (G01)-(G06) are real-value functions.
From Proposition 7, we have

$$\tilde{C}_{1\tilde{A}_j}(\cdot) = (2-j)\,\beta_2\,(\rho\sigma_\delta + \sigma_\delta^2 B(\tau^*;\theta)) + \alpha(T_c - \tau) + \lambda_\delta\beta_2,$$

$$\tilde{C}_{2\tilde{A}_j}(\cdot) = \phi\beta_2\,(\rho\sigma_\delta + \sigma_\delta^2 B(\tau^*;\theta)),$$

$$\tilde{C}_{1\tilde{B}_j}(\cdot) = (2-j)\beta_1(\rho\sigma_\delta + \sigma_\delta^2 B(\tau^*;\theta)) + \lambda_\delta\beta_1 - \kappa,$$

$$\tilde{C}_{2\tilde{B}_j}(\cdot) = \phi\beta_1(\rho\sigma_\delta + \sigma_\delta^2 B(\tau^*;\theta)),$$

$$\tilde{D}_{1\tilde{A}_j}(\cdot) = -\tfrac{1}{2}\phi^2\beta_2\,(1 + 2\rho\sigma_\delta B(\tau^*;\theta) + \sigma_\delta^2 B^2(\tau^*;\theta)),$$

$$\tilde{D}_{2\tilde{A}_j}(\cdot) = -\tfrac{1}{2}(-1)^j\phi\beta_2(1 + 2\rho\sigma_\delta B(\tau^*;\theta) + \sigma_\delta^2 B^2(\tau^*;\theta)),$$

$$\tilde{D}_{1\tilde{B}_j}(\cdot) = -\tfrac{1}{2}\phi^2\beta_1(1 + 2\rho\sigma_\delta B(\tau^*;\theta) + \sigma_\delta^2 B^2(\tau^*;\theta)),$$

$$\tilde{D}_{2\tilde{B}_j}(\cdot) = -\tfrac{1}{2}(-1)^j\phi\beta_1(1 + 2\rho\sigma_\delta B(\tau^*;\theta) + \sigma_\delta^2 B^2(\tau^*;\theta)),$$

$\tilde{A}_{kj}(\cdot)$ and $\tilde{B}_{kj}(\cdot)$, $k = 1,2,$ are unknown functions, for $j = 1,2,$ where $\tau^* = T - T_c + \tau.$
Plugging (G01)-(G06) into Equation (1.3.33), we obtain the following systems of ODEs:

$$\left(\tilde{A}'_{1j}+\tilde{A}'_{2j}\mathbb{i}\right)-\tfrac{1}{2}\beta_2\sigma_\delta^2\left(\tilde{B}_{1j}+\tilde{B}_{2j}\mathbb{i}\right)^2-\left(\tilde{C}_{1\tilde{A}_j}+\tilde{C}_{2\tilde{A}_j}\mathbb{i}\right)\left(\tilde{B}_{1j}+\tilde{B}_{2j}\mathbb{i}\right)-\left(\tilde{D}_{1\tilde{A}_j}+\tilde{D}_{2\tilde{A}_j}\mathbb{i}\right)=0$$

$$\left(\tilde{B}'_{1j}+\tilde{B}'_{2j}\mathbb{i}\right)-\tfrac{1}{2}\beta_1\sigma_\delta^2\left(\tilde{B}_{1j}+\tilde{B}_{2j}\mathbb{i}\right)^2-\left(\tilde{C}_{1\tilde{B}_j}+\tilde{C}_{2\tilde{B}_j}\mathbb{i}\right)\left(\tilde{B}_{1j}+\tilde{B}_{2j}\mathbb{i}\right)-\left(\tilde{D}_{1\tilde{B}_j}+\tilde{D}_{2\tilde{B}_j}\mathbb{i}\right)=0$$

for $j=1,2,$ where $\,' = \dfrac{d}{d\tau}$. (G07)

One can calculate

$$\left(\tilde{B}_{1j}+\tilde{B}_{2j}\mathbb{i}\right)^2=\left(\tilde{B}_{1j}^2-\tilde{B}_{2j}^2\right)+2\tilde{B}_{1j}\tilde{B}_{2j}\mathbb{i},$$

$$\left(\tilde{C}_{1\tilde{A}_j}+\tilde{C}_{2\tilde{A}_j}\mathbb{i}\right)\left(\tilde{B}_{1j}+\tilde{B}_{2j}\mathbb{i}\right)=\left(\tilde{C}_{1\tilde{A}_j}\tilde{B}_{1j}-\tilde{C}_{2\tilde{A}_j}\tilde{B}_{2j}\right)+\left(\tilde{C}_{1\tilde{A}_j}\tilde{B}_{2j}+\tilde{C}_{2\tilde{A}_j}\tilde{B}_{1j}\right)\mathbb{i},$$

$$\left(\tilde{C}_{1\tilde{B}_j}+\tilde{C}_{2\tilde{B}_j}\mathbb{i}\right)\left(\tilde{B}_{1j}+\tilde{B}_{2j}\mathbb{i}\right)=\left(\tilde{C}_{1\tilde{B}_j}\tilde{B}_{1j}-\tilde{C}_{2\tilde{B}_j}\tilde{B}_{2j}\right)+\left(\tilde{C}_{1\tilde{B}_j}\tilde{B}_{2j}+\tilde{C}_{2\tilde{B}_j}\tilde{B}_{1j}\right)\mathbb{i}.$$

Putting the above results into (G07) and, for each equation, equating the real part and the imaginary part to zero give us the following systems of ODEs:

$$\begin{aligned}
\tilde{A}'_{1j}-\tfrac{1}{2}\beta_2\sigma_\delta^2\left(\tilde{B}_{1j}^2-\tilde{B}_{2j}^2\right)-\left(\tilde{C}_{1\tilde{A}_j}\tilde{B}_{1j}-\tilde{C}_{2\tilde{A}_j}\tilde{B}_{2j}\right)-\tilde{D}_{1\tilde{A}_j}&=0,\\
\tilde{A}'_{2j}-\beta_2\sigma_\delta^2\tilde{B}_{1j}\tilde{B}_{2j}-\left(\tilde{C}_{1\tilde{A}_j}\tilde{B}_{2j}+\tilde{C}_{2\tilde{A}_j}\tilde{B}_{1j}\right)-\tilde{D}_{2\tilde{A}_j}&=0,\\
\tilde{B}'_{1j}-\tfrac{1}{2}\beta_1\sigma_\delta^2\left(\tilde{B}_{1j}^2-\tilde{B}_{2j}^2\right)-\left(\tilde{C}_{1\tilde{B}_j}\tilde{B}_{1j}-\tilde{C}_{2\tilde{B}_j}\tilde{B}_{2j}\right)-\tilde{D}_{1\tilde{B}_j}&=0,\\
\tilde{B}'_{2j}-\beta_1\sigma_\delta^2\tilde{B}_{1j}\tilde{B}_{2j}-\left(\tilde{C}_{1\tilde{B}_j}\tilde{B}_{2j}+\tilde{C}_{2\tilde{B}_j}\tilde{B}_{1j}\right)-\tilde{D}_{2\tilde{B}_j}&=0,
\end{aligned} \tag{G08}$$

for $j=1,2,$ subject to the initial conditions

$$\tilde{A}_{kj}(\phi;0,T_c,T;\theta)=\tilde{B}_{kj}(\phi;0,T_c,T;\theta)=0,\quad \text{for } k=1,2. \tag{G09}$$

For given a parameter vector θ, maturity dates T_c, T, and a current time t, one can apply some traditional numerical schemes such as Runge-Kutta methods for solving the systems (G08) subject to the initial conditions (G09) to obtain the values of the unknown functions at any point $\phi\in(0,\infty)$.

Appendix H

Sensitivity Analysis

In this appendix, we investigate the sensitivities of the parameter $\lambda \equiv \lambda_S$ to the following extraction formulas of spot prices, convenience yields, and futures prices, i.e.,

$$S(t,F_t^{T_1}(\lambda),F_t^{T_2}(\lambda);\lambda) = \exp\left(\frac{B(T_1-t;\lambda)\ln F_t^{T_2}(\lambda) - B(T_2-t;\lambda)\ln F_t^{T_1}(\lambda) + G(t,T_1,T_2;\lambda)}{B(T_1-t;\lambda) - B(T_2-t;\lambda)}\right) \tag{H01}$$

$$\delta(t,F_t^{T_1}(\lambda),F_t^{T_2}(\lambda);\lambda) = \frac{\left(\ln F_t^{T_1}(\lambda) - F_t^{T_2}(\lambda)\right) + \left(A(T_2-t;\lambda) - A(T_1-t;\lambda)\right)}{B(T_1-t;\lambda) - B(T_2-t;\lambda)} \tag{H02}$$

$$F^T(t,F_t^{T_1}(\lambda),F_t^{T_2}(\lambda);\lambda) = S(t,F_t^{T_1}(\lambda),F_t^{T_2}(\lambda);\lambda)\, e^{A(T-t;\lambda)+B(T-t;\lambda)\delta(t,F_t^{T_1}(\lambda),F_t^{T_2}(\lambda);\lambda)} \tag{H03}$$

where

$$G(t,T_1,T_2;\lambda) = A(T_1-t;\lambda)B(T_2-t;\lambda) - A(T_2-t;\lambda)B(T_1-t;\lambda)$$

and the functions $A(\tau;\lambda)$ and $B(\tau;\lambda)$ are given in Proposition 5.

Under this investigation, we assume that the model parameters except λ_S are set to be equal to their estimates as tabulated in Table 3.4. We measure the sensitivities by using local sensitivity analysis (LSA). In other words, we consider the first derivatives of S, δ, and F^T with respect to the parameter λ at the observed time points $t_n, n = 1,\ldots,N$. The first derivatives are evaluated at (λ_S^A, t_n) for all n where λ_S^A is the estimate of λ_S as given in Table 3.4. Firstly, we define the sensitivities in terms of the following quantities:

$$\mathcal{S}_S^n := (dS/d\lambda)\big|_{\lambda=\lambda_S^A, t=t_n}, \tag{H04}$$

$$\mathcal{S}_\delta^n := (d\delta/d\lambda)\big|_{\lambda=\lambda_S^A, t=t_n}, \tag{H05}$$

$$\mathcal{S}_{F^T}^n := (dF^T/d\lambda)\big|_{\lambda=\lambda_S^A, t=t_n}, \tag{H06}$$

for $n = 1,\ldots,N$. The quantities defined in (H04)-(H06) are, respectively, the local sensitivity indexes measuring the effects on S, δ, and F^T of perturbing λ around the estimate of λ_S.

We start by computing the above first derivatives:

$$dS/d\lambda = S\left(L\frac{dU_S}{d\lambda} - U_S\frac{dL}{d\lambda}\right)/L^2, \tag{H07}$$

$$d\delta / d\lambda = \left(L \frac{dU_\delta}{d\lambda} - U_\delta \frac{dL}{d\lambda} \right) / L^2, \tag{H08}$$

$$dF^T / d\lambda = F^T \left(\frac{1}{S} \frac{dS}{d\lambda} + \frac{d}{d\lambda} A(T-t;\lambda) + B(T-t;\lambda) \frac{d\delta}{d\lambda} + \delta \frac{d}{d\lambda} B(T-t;\lambda) \right), \tag{H09}$$

where

$$L \equiv L(t, T_1, T_2; \lambda) := B(T_1 - t; \lambda) - B(T_2 - t; \lambda),$$

$$U_S \equiv U_S(t, F_t^{T_1}(\lambda), F_t^{T_2}(\lambda); \lambda) := (B(T_1 - t; \lambda) \ln F_t^{T_2} - B(T_2 - t; \lambda) \ln F_t^{T_1}) + G(t, T_1, T_2; \lambda),$$

$$U_\delta \equiv U_\delta(t, F_t^{T_1}(\lambda), F_t^{T_2}(\lambda); \lambda) := (\ln F_t^{T_1} - \ln F_t^{T_2}) + (A(T_2 - t; \lambda) - A(T_1 - t; \lambda)),$$

$$\frac{dU_S}{d\lambda} = \left(\frac{B(T_1 - t;\lambda)}{F_t^{T_2}(\lambda)} \frac{dF_t^{T_2}}{d\lambda} + \ln F_t^{T_2} \frac{d}{d\lambda} B(T_1 - t;\lambda) \right) - \left(\frac{B(T_2 - t;\lambda)}{F_t^{T_1}(\lambda)} \frac{dF_t^{T_1}}{d\lambda} + \ln F_t^{T_1} \frac{d}{d\lambda} B(T_2 - t;\lambda) \right) + \frac{dG}{d\lambda},$$

$$\frac{dU_\delta}{d\lambda} = \left(\frac{1}{F_t^{T_1}(\lambda)} \frac{dF_t^{T_1}}{d\lambda} - \frac{1}{F_t^{T_2}(\lambda)} \frac{dF_t^{T_2}}{d\lambda} \right) + \frac{d}{d\lambda} \left(A(T_2 - t;\lambda) - A(T_1 - t;\lambda) \right).$$

We approximate the first derivatives of $F_t^{T_i}(\lambda), i = 1,2$ with respect to λ contained in the above formulas of $\frac{dU_S}{d\lambda}$ and $\frac{dU_\delta}{d\lambda}$ by using the finite-difference approximations, i.e.,

$$\frac{dF_t^{T_i}(\lambda)}{d\lambda} \approx \frac{F_t^{T_i}(t, F_t^{T_1}(\lambda + h), F_t^{T_2}(\lambda + h); \lambda + h) - F_t^{T_i}(t, F_t^{T_1}(\lambda), F_t^{T_2}(\lambda); \lambda)}{h},$$

for $i = 1,2$, where $h = 10^{-4}$ is chosen for this investigation and we have approximated $F_t^{T_i}(\lambda + h)$ by $F_t^{T_i}(\lambda)$ for $i = 1,2$. Applying the futures prices data of WR5 and RSRS3, $F_{N_w}^{T_w}$ and $F_{N_s}^{T_s}$, to (H07)-(H09), we then obtain the quantities (H04)-(H06) as desired.

We define further prediction errors of spot prices, convenience yields, and futures prices as follows. On an observed day t_n, the prediction errors of the spot price, the convenience yield, and the futures price are defined as follows:

$$\mathcal{E}_S^n := \pm \left| \left[S(t_n, F_{t_n}^{T_1^n}(\lambda), F_{t_n}^{T_2^n}(\lambda); \lambda \right]_{\lambda = \lambda_S^A}^{\lambda = \lambda_S^A + \Delta\lambda} \right| \approx \pm \left| \mathcal{S}_S^n \right| \Delta\lambda, \tag{H10}$$

$$\mathcal{E}_\delta^n := \pm \left| \left[\delta(t_n, F_{t_n}^{T_1^n}(\lambda), F_{t_n}^{T_2^n}(\lambda); \lambda \right]_{\lambda = \lambda_S^A}^{\lambda = \lambda_S^A + \Delta\lambda} \right| \approx \pm \left| \mathcal{S}_\delta^n \right| \Delta\lambda, \tag{H11}$$

$$\mathcal{E}_{F^T}^n := \pm \left| \left[F^T(t_n, F_{t_n}^{T_1^n}(\lambda), F_{t_n}^{T_2^n}(\lambda); \lambda \right]_{\lambda = \lambda_S^A}^{\lambda = \lambda_S^A + \Delta\lambda} \right| \approx \pm \left| \mathcal{S}_{F^T}^n \right| \Delta\lambda, \tag{H12}$$

where $\Delta\lambda = 2s$, s is the corresponding approximate asymptotic standard deviation of the MLE of λ_S tabulated in Table 3.4.

In Figures H1-H3, we illustrate the sensitivities $\mathcal{S}_S^n$, $\mathcal{S}_\delta^n$, and $\mathcal{S}_{F^T}^n$ together with the prediction errors $\mathcal{E}_S^n$, $\mathcal{E}_\delta^n$, and $\mathcal{E}_{F^T}^n$ in WR5 Case during the sample period. In RSRS3 case, the corresponding quantities are illustrated in Figures H4-H6 during the sample period. The results obtained show that the sensitivities $\mathcal{S}_S^n$ and $\mathcal{S}_\delta^n$ in WR5 and RSRS3 cases are small during the sample periods. These results imply that the variations of λ_S within the corresponding confident intervals tabulated in Table 3.4 of Chapter 3 do not affect much the extracted values of the spot prices and the convenience yields of WR5 and RSRS3. The average of $\mathcal{E}_S^n$ in WR5 case during the sample period is ± 0.001 Bahts/Kg. This means that if the true MLE of λ_S is known, the extracted spot prices of WR5 obtained by inserting λ_S into the extraction formula (H01) should belong within the range:

$$[S(t;\lambda_S^A) - 0.001, S(t;\lambda_S^A) + 0.001] \text{ for all observed day } t.$$

The same procedure can be applied to the remaining average prediction errors of $\mathcal{E}_S^n$ and $\mathcal{E}_\delta^n$ during the sample periods to obtain the corresponding ranges. As illustrated in Figures H3 and H6, the sensitivities $\mathcal{S}_{F^T}^n$ in WR5 and RSRS3 cases approach zero as the observed times approach the maturity dates. The sensitivities $\mathcal{S}_{F^T}^n$ in WR5 case are small during its sample period, while the sensitivities $\mathcal{S}_{F^T}^n$ in RSRS3 case are quite large at the beginning of its sample period. Consequently, the prediction errors $\mathcal{E}_{F^T}^n$ in RSRS3 case are high in that observed days.

It can be noticed from the graphs in Figure H1 for WR5 case that the sensitivities exhibit several jumps on the observed days which are close to the beginning of the maturity months. Consequently, the prediction errors are high on the observed days. This is similar to RSRS3 case which can be noticed from the graphs in Figure H4. These results come from the futures prices data of WR5 and RSRS3 and can be explained as follows. We started collecting the futures prices $F_t^{T_1}(\lambda)$ and $F_t^{T_2}(\lambda)$ from the first trading day $t = 0$ where T_1 and T_2 are the closest and the second-closest maturity dates of the futures contracts that had been traded on day t. When the first contract expired at the beginning of its maturity month, we started collecting the futures prices of the other contract in which its maturity date, say T_3, is closest to T_2. This might cause the price jumps happened because Assumption A might be violated if T_3 is far from the trading day t. In other words, the observed futures prices are not close to the exact no-arbitrage futures prices on the trading days which are far from the maturity date of the futures contract. These affects can also be noticed from the graphs in Figures H2 and H5 for the convenience yields and from the graphs in Figures H3 and H6 for the futures prices.

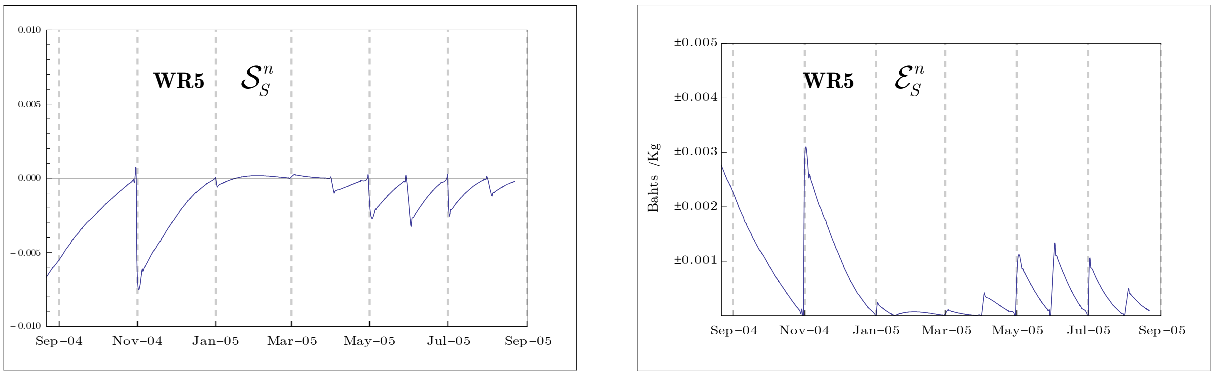


Figure H1: The sensitivity of the parameter λ_S to the extracted spot prices of WR5 and the prediction errors of the spot prices during the sample period. The average of $\mathcal{E}_S^n$ during the sample period is ± 0.001 Bahts/Kg.

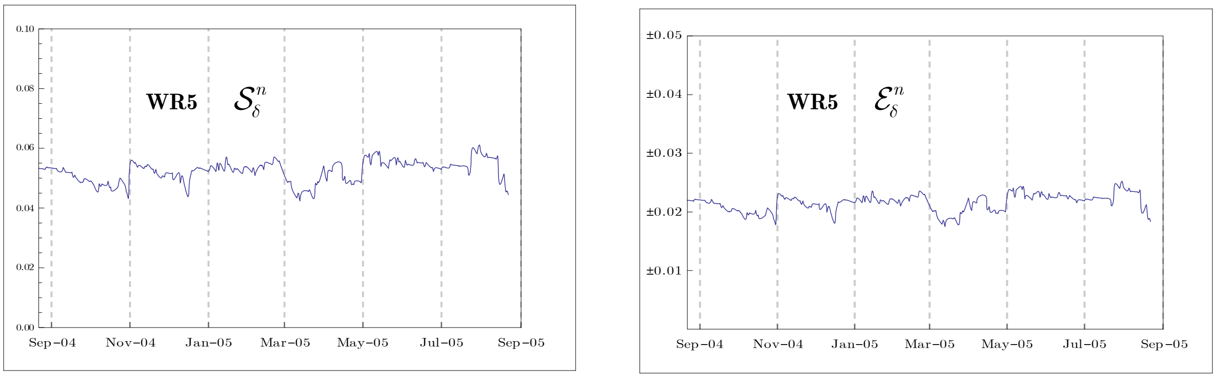


Figure H2: The sensitivity of the parameter λ_S to the extracted convenience yields of WR5 and the prediction errors of the convenience yields during the sample period. The average of $\mathcal{E}_\delta^n$ during the sample period is ± 0.022.

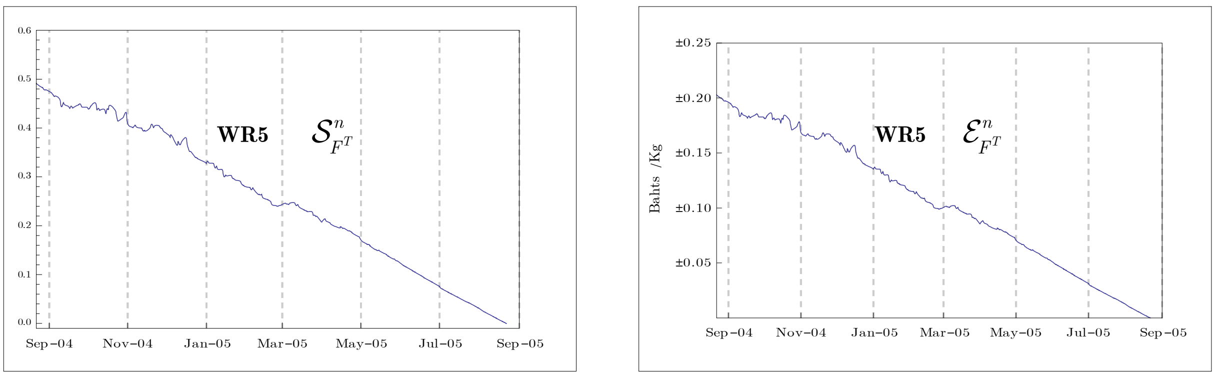


Figure H3: The sensitivity of the parameter λ_S to the extracted futures prices of WR5 with maturity $T = 1$ and the prediction errors of the futures prices during the sample period. The average of $\mathcal{E}_{F^T}^n$ during the sample period is ± 0.107 Bahts/Kg.

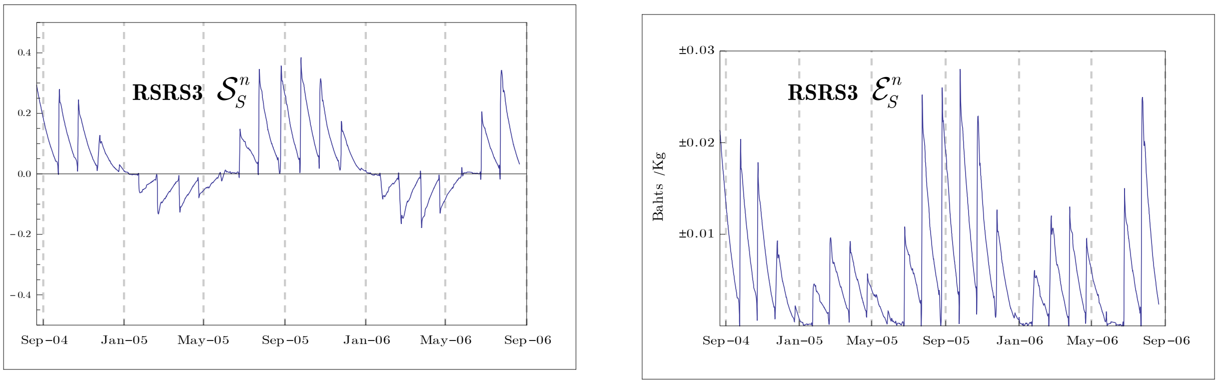


Figure H4: The sensitivity of the parameter λ_S to the extracted spot prices of RSRS3 and the prediction errors of the spot prices during the sample period. The average of $\mathcal{E}_S^n$ during the sample period is ± 0.060 Bahts/Kg.

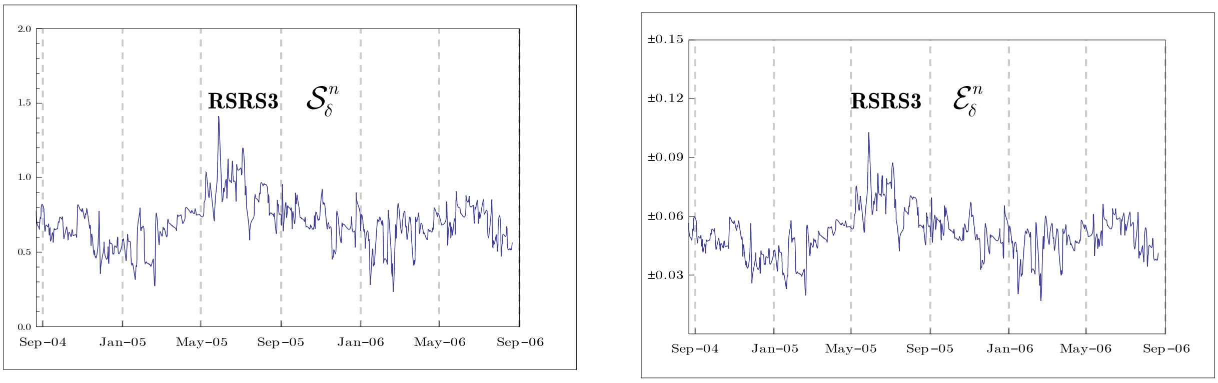


Figure H5: The sensitivity of the parameter λ_S to the extracted convenience yields of RSRS3 and the prediction errors of the convenience yields during the sample period. The average of $\mathcal{E}_\delta^n$ during the sample period is ± 0.051.

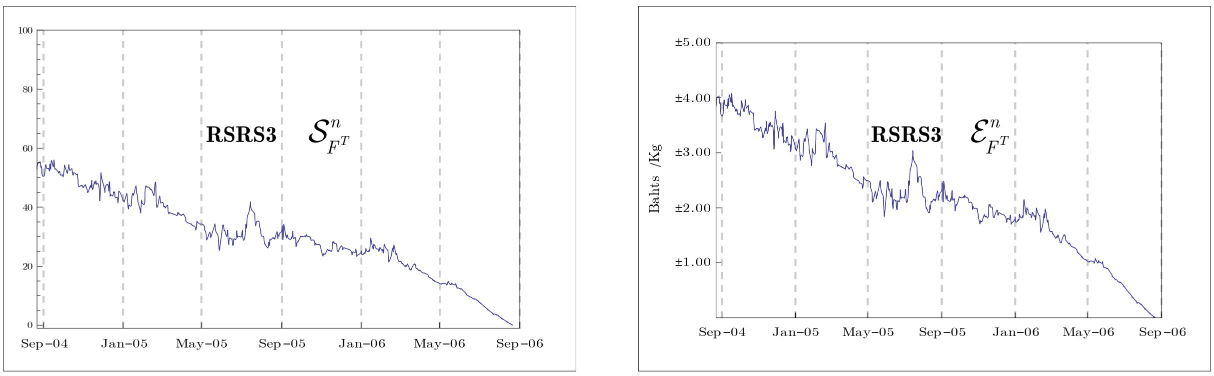


Figure H6: The sensitivity of the parameter λ_S to the extracted futures prices of RSRS3 with maturity $T = 2$ and the prediction errors of the futures prices during the sample period. The average of $\mathcal{E}_{F^T}^n$ during the sample period is ± 2.223 Bahts/Kg.

Definitions

Commodity Derivatives

A *derivative security* (more briefly, *derivative*) of an underlying asset is a financial contract whose value at a future date T is determined exactly by the market prices of the underlying asset within the time interval $[0,T]$. If the underlying asset is referred to a commodity then the derivative security is known as the *commodity derivative*.

Forwards and Futures

A *forward contract* (more briefly, *forward*) of an underlying asset is a derivative security giving one has an obligation to a specified transaction of the underlying asset, at a certain future time for a certain price. The certain future time is known as the *expiration date* or *maturity* of the contract and the certain price is known as the *forward price*. The expiration date of the contract and the forward price are written when the contract is entered by two parties. After that the forward price is known as the *delivery price*. Like a forward contract, a *futures contract* (more briefly, *futures*) of an underlying asset is a forward contract that is traded on an exchange. The exchange specifies certain standardized features of the contract and provides a mechanism that gives the two parties a guarantee that the contract will be honored. Such the exchange is known as the *futures market* and the forward price is known as the *futures price*.

Futures Options

An *option contract* (more briefly, *option*) is a derivative security giving one the right but not obligation to make a specified transaction of the underlying asset at a future date at a certain price. The futures date is known as the *expiration date* or *maturity* of the option and the specified price is known as the *exercise price* or *strike price*. *Call* options give one the right to buy. *Put* options give one the right to sell. *European options* give one the right to *exercise* the options only on the expiration date. If the underlying asset is referred to a futures contract then the European options are known as the *European Futures Option*s.

Assumptions

The No-Arbitrage Assumptions

(1) The market is *arbitrage-free*, that is, for any portfolio $\varphi = (\varphi_t)$,

$$V_\varphi(0) = 0 \text{ and } V_\varphi(T) \geq 0, \ \mathbb{P}\text{- a.s. for all time } T > 0 \text{ imply } V_\varphi(T) = 0, \ \mathbb{P}\text{- a.s.},$$

where $V_\varphi(t) \equiv V_\varphi(t, S_t, \delta_t, \varphi_t)$ denotes the value of the portfolio φ at time t and $\mathbb{P}$ denotes an original probability measure. Namely, if a portfolio requires a null investment and is riskless (there is no possible loss at the time horizon T), then its terminal value at time T has to be zero.

(2) The market participants are subject to no transaction costs when they trade.

(3) The market participants are subject to no tax rate on all net trading profits.

(4) The market participants can borrow/lend money at the same risk free rate of interest.

Under the no-arbitrage assumptions, the fair-prices (or the no-arbitrage prices) of futures and options contracts can be determined under a so-called equivalent martingale measure (or the risk-neutral probability measure) $\mathbb{Q} \sim \mathbb{P}$.

Remark: The assumptions (2)-(4) are known as *"the market is frictionless"*.

Assumption A

In a futures market of a commodity, for every trading day *t*, we can observe two no-arbitrage futures prices $F_t^{T_1}$ and $F_t^{T_2}$ without measurement error where T_1 and T_2 are, respectively, the closest and the second-closest maturity dates of the corresponding futures contracts that have been traded on day *t*.

Acronyms

AFET	Agricultural Futures Exchange of Thailand
ANRPC	Association Natural Rubber Producing Countries
CIR	Cox-Ingersoll-Ross
DE	Differential Evolution
GBM	Geometric Brownian Motion
LAMN	Locally Asymptotically Mixed Normal
LHS	Left Hand Side
Log-	Logarithmic
MLE	Maximum Likelihood Estimator
NR	Natural Rubber
ODE	Ordinary Differential Equation
OU	Ornstein-Uhlenbeck
PDE	Partial Differential Equation
RHS	Right Hand Side
RSRS3	Ribbed Smoked Rubber Sheet no.3
SDE	Stochastic Differential Equation
USDA	United States Department of Agriculture
WR5	White Rice 5%

List of Selected Symbols

Symbol	Meaning	Defined in
S_t	commodity spot price at a current time t	Intro.
δ_t	instantaneous convenience yield at a current time t	Intro.
$\mathbb{P}$	original probability measure	Intro.
$\mathbb{Q}$	equivalent martingale measure or risk-neutral probability measure	Intro.
$\mathbb{Q} \sim \mathbb{P}$	measure $\mathbb{Q}$ is equivalent to measure $\mathbb{P}$	Intro.
F_t^T	futures price at date t of a futures contract having maturity date T	Sec. 1.1.1
$\alpha_T(t)$	seasonal function	Sec. 1.2.1
r	risk free interest rate	Sec. 1.2.1
W_t	Brownian motion or Wiener process	Sec. 1.2.1
Ω	sample space	Sec. 1.2.1
$\mathcal{F}$	sigma algebra	Sec. 1.2.1
$(\mathcal{F}_t)_{t\geq 0}$	filtration	Sec. 1.2.1
θ	vector of unknown model parameters: $(\beta_1, \beta_2, \kappa, \sigma_\delta, \lambda_S, \lambda_\delta, \rho, \alpha_0, \alpha_1^{(k)}, \alpha_2^{(k)}), k = 1,..., K^\alpha$	Sec. 1.2.1
K^α	number of terms in the sum of seasonal function $\alpha_T(t)$	Sec. 1.2.1
$E_{\mathbb{Q}}[\cdot]$	expectation with respect to the probability measure $\mathbb{Q}$	Sec. 1.2.3
$\mathrm{Var}_{\mathbb{Q}}[\cdot]$	variance with respect to the probability measure $\mathbb{Q}$	Sec. 1.2.3
A^{T}	transpose of matrix A	Sec. 1.3.1
$[f(x)]_{x=a}^{x=b}$	$f(b) - f(a)$	Sec. 1.3.2
X_t^T	logarithm of F_t^T	Sec. 1.3.4
$\tilde{p}_{X^T}$	forward transition density of the process X_t^T	Sec. 2.1
$\hat{\delta}(\cdot)$	Dirac-delta function on $\mathbb{R}$	Sec. 2.1
$a_T(\cdot)$	diffusion coefficient of the process X_t^T	Sec. 2.1
$\tilde{a}_T(t;\mathbf{s})$	$a_T(t) + a_T(\mathbf{s})$	Sec. 2.1
$\tilde{p}_{X^T}^A(\cdot)$	approximate forward transition density of the process X_t^T	Sec. 2.1

Symbol	Meaning	Defined in
X_N^T	logarithmic futures prices data	Sec. 2.2
$\tilde{l}_{X^T}(\cdot)$	logarithm of $\tilde{p}_{X^T}(\cdot)$	Sec. 2.2
$\tilde{l}_N(\cdot)$	log-likelihood function of X_N^T	Sec. 2.2
$\tilde{l}_{X^T}^A(\cdot)$	logarithm of $\tilde{p}_{X^T}^A(\cdot)$	Sec. 2.2
$\tilde{l}_N^A(\cdot)$	approximate log-likelihood function of X_N^T	Sec. 2.2
Θ	parameter space	Sec. 2.2
n_θ	number of unknown parameters	Sec. 2.2
Θ°	set of interior points of Θ	Sec. 2.2
θ_0	vector of true-parameters	Sec. 2.2
$C^3(\Theta)$	set of three-time continuously differentiable functions on Θ	Sec. 2.2
$\tilde{\theta}_{N,X_N^T}^{MLE}$	MLE given X_N^T	Sec. 2.2
$\tilde{\theta}_{N,X_N^T,\Delta}^A$	approximate MLE given X_N^T and Δ	Sec. 2.2
$F_{N_w}^{T_w}$	no-arbitrage futures prices data of WR5	Sec. 3.2
$F_{N_s}^{T_s}$	no-arbitrage futures prices data of RSRS3	Sec. 3.2
$n_{\mathbf{F}}^{\bar{T}}$	number of days in which the commodity **F** having maturity date $\bar{T}$ had been traded during the sample period	Sec. 3.6.1
$n_{\mathbf{F}}^{\bar{T},1}$	number of days in which we do not use the futures prices of the commodity **F** having maturity date $\bar{T}$ as the observed data in estimation of the model parameters	Sec. 3.6.1
$n_{\mathbf{F}}^{\bar{T},2}$	number of days in which we use the futures prices of the commodity **F** having maturity date $\bar{T}$ as the observed data in estimation of the model parameters	Sec. 3.6.1
$\bar{D}_{\mathbf{F}}^{\bar{T}}$	average of percentage absolute price difference between the observed and the predicted futures prices of commodity **F** having maturity date $\bar{T}$	Sec. 3.6.1
$S_{\mathbf{F}}^{\bar{T}}$	sample standard deviation of the percentage absolute price differences between the observed and the predicted futures prices of commodity **F** having maturity date $\bar{T}$	Sec. 3.6.1
$C_{\mathbf{F}}^{\bar{T}}$	correlation between the observed and the predicted futures prices of commodity **F** having maturity date $\bar{T}$	Sec. 3.6.1

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